

# THE STRUCTURE OF VERMA MODULES OVER THE $N = 2$ SUPERCONFORMAL ALGEBRA

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We classify degeneration patterns of Verma modules over the  $N = 2$  superconformal algebra in two dimensions. Explicit formulae are given for singular vectors that generate maximal submodules in each of the degenerate cases. The mappings between Verma modules defined by these singular vectors are *embeddings*; in particular, their compositions never vanish. As a by-product, we also obtain general formulae for  $N = 2$  subsingular vectors.

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## 1 Introduction

In this paper, we describe the structure of submodules and singular vectors in Verma modules over the  $N = 2$  superconformal algebra in two dimensions — the  $N = 2$  supersymmetric extension of the Virasoro algebra [A]. This algebra underlies the construction of  $N = 2$  strings [A, M, FT, OV] (with its possible role in the M-theory proposed in [KM]), and, on the other hand, is realized on the world-sheet of any non-critical string theory [GRS, BLNW]. Other (non-exhaustive) references on the  $N = 2$  superconformal algebra and  $N = 2$  models in conformal field theory and string theory are [BFK, EHY, SS, DVPYZ, G, IK, KS, EG, Ga].

The  $N = 2$  algebra, however, hasn't been very privileged in several respects, first of all because it is not an affine Lie algebra. It does not admit a root system enjoying all the properties of root systems of affine Lie algebras, hence, in particular, there is no canonical triangular decomposition. As a result, there is no canonical way to impose 'highest-weight'-type conditions on a vacuum vector (hence, on singular vectors) in representations of the algebra. Trying to follow the *formal* analogy with the case of affine Lie algebras (e.g.,  $\widehat{sl}(2)$ ) and imitating the highest-weight conditions imposed there leads to several complications, if not inconsistencies, with the definition and properties of  $N = 2$  Verma modules. These complications are related to the fact that there exist two different types of Verma-like modules, and, while modules of one

type can be submodules of the other, the converse is not true. The definition of singular vectors carried over from the affine Lie-algebra case does not distinguish between the two types of submodules.

Among other facts pertaining to the  $N=2$  algebra, let us note that the different *sectors* of the algebra (the Neveu–Schwarz and Ramond ones) are isomorphic, which is in contrast to the  $N=1$  case. This is due to the  $N=2$  spectral flow [SS]. Thus, there is ‘the only’  $N=2$  algebra<sup>1</sup> and its isomorphic images under the spectral flow. However, the basis in the algebra can be chosen in different ways, since the presence of the  $U(1)$  current allows one to change the energy-momentum tensor by the derivative of the current; the algebras that appear in different contexts are in fact isomorphic [EY, W] to one and the same  $N=2$  algebra. Finally, even the terminology used in the  $N=2$  representation theory does not appear to be unified, which may again be related to the fact that the situation which is familiar from the affine Lie algebras does not literally carry over to  $N=2$ .

The object of our study is possible degenerations (reducibility patterns) of  $N=2$  Verma modules, i.e., the structure of submodules in these modules. This is more involved than in more familiar cases of the Virasoro algebra and the standard Verma modules over the affine algebra  $\widehat{\mathfrak{sl}}(2)$ , due to two main reasons. First, the  $N=2$  algebra has rank 3, which gives its modules more possibilities to degenerate. Second, as we have already mentioned, there are two different types of  $N=2$  Verma-like modules that have to be distinguished clearly; we follow refs. [ST2, FST] in calling them the *topological* and *massive* Verma modules (in a different terminology, the first ones are chiral, while the second ones are tacitly understood to be ‘*the*’  $N=2$  Verma modules). The topological Verma modules appearing as submodules are *twisted*, i.e., transformed by the spectral flow. Ignoring the existence of two types of modules and trying to describe degenerations of  $N=2$  Verma modules in terms of only massive Verma (sub)modules results in an incorrect picture, e.g. apparent relations in Verma modules would then seem to exist, in contradiction with the definition of *Verma* modules. In the embedding diagrams known in the literature, similarly, some singular vectors appear to vanish when constructed in a module built on another singular vector — in which case one can hardly talk about *embedding* diagrams. The actual situation is that topological Verma submodules may exist in massive Verma modules, with one extra annihilation condition being imposed on the highest-weight vector of such submodules. For example, submodules generated by the so-called charged singular vectors [BFK] are *always* (twisted) topological. That this fundamental fact about the charged singular vectors has not been widely appreciated, is because the nature of submodules is obscured when one employs singular vectors defined using a formal analogy with the case of affine Lie algebras. In fact, any mapping from a massive Verma module into a topological Verma module necessarily has a kernel that contains another topological Verma submodule, which makes a sequence of such mappings look more like a BGG-resolution rather than an embedding diagram. Another reason why literally copying the definition of singular vectors from the affine Lie algebras complicates the analysis of  $N=2$  modules is that using such singular vectors entails *subsingular* vectors.

Generally, when considering representations of algebras of rank  $\geq 3$ , one should take care of whether a given singular vector or a set of singular vectors generate a *maximal* submodule. That a submodule  $\mathcal{U}_2$  generated from all singular vectors in a Verma module  $\mathcal{U}$  is not maximal means that there exists a proper submodule  $\mathcal{U}_1 \neq \mathcal{U}_2$  such that  $\mathcal{U}_2 \subset \mathcal{U}_1 \subset \mathcal{U}$ . Then, the quotient module  $\mathcal{U}/\mathcal{U}_2$  contains a submodule, which in the simplest case would be generated from one or several singular vectors (otherwise, the story repeats). However, these vectors are *not* singular in  $\mathcal{U}$ , i.e., before taking the quotient with respect to  $\mathcal{U}_2$ . They are commonly known as subsingular vectors.

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<sup>1</sup>With the exception of a somewhat exotic ‘twisted sector’, where one of the fermions has half-integer, and the other, integer, modes, *which we do not touch upon in this paper*.

The question of whether or not singular vector(s) generate a maximal submodule is unambiguous for the affine Lie algebras, where the root system determines a fixed set of operators from the algebra that are required to annihilate a state in order that it be a singular vector. As we have already remarked, such annihilation conditions are not defined uniquely in the  $N=2$  case. Since the significance of singular vectors consists in providing a description of submodules, one should thus focus one's attention on the structure of submodules in  $N=2$  Verma modules. As regards singular vectors, then, one has two options: either to fix the convention that singular vectors satisfy *some* annihilation conditions (for instance, those copied literally from the case of affine Lie algebras) and then to find a system of sub-, subsub-, ...-singular vectors that 'compensate' for the failure of the chosen singular vectors to generate maximal submodules; or to try to define singular vectors in such a way that they generate maximal submodules, in which case the structure of submodules would be described without introducing subsingular vectors.

In what follows, we present a regular way to single out and to explicitly construct those vectors that generate maximal submodules in  $N=2$  Verma modules<sup>2</sup>. They turn out to satisfy 'twisted' annihilation conditions, i.e. those given by a spectral flow transform [SS, LVW] of the annihilation conditions imposed on the highest-weight vector in the module. With this definition of singular vectors, subsingular vectors become redundant. However, in view of controversial statements that have been made regarding 'subsingular vectors in  $N=2$  Verma modules' [D, GRR, EG], we will also show how our description can be adapted to provide a systematic way to construct the states in  $N=2$  Verma modules that are subsingular vectors once singular vectors are defined by the conventional, 'untwisted', annihilation conditions. The general picture that emerges in this way is very simple and can be outlined as follows.

Recall that, in modules over a  $\mathbb{Z} \times \mathbb{Z}$ -graded algebra, any vector that satisfies 'highest-weight' conditions is a member of the family of extremal states which make up an *extremal diagram*<sup>3</sup> (see Eqs. (2.6) and (2.9) for the topological and the massive Verma modules, respectively). For the  $N=2$  algebra, a given singular vector that we consider satisfies twisted highest-weight conditions with a certain integral twist  $\theta$ ; this vector belongs to the extremal diagram that consists of states satisfying twisted highest-weight conditions with all integral twists. That of the extremal states which satisfies the conventional, 'untwisted', highest-weight conditions is the conventional singular vector. Now, it is the properties of the extremal diagram that are responsible for whether or not all of the extremal states generate the same submodule. Generically, it is irrelevant which of the representatives of the extremal diagram is singled out as '*the*' singular vector. In the degenerate cases, however, there do exist preferred representatives that generate the maximal possible submodule, while other extremal states generate a smaller submodule. Moreover, there exists a systematic way to divide  $N=2$  extremal diagrams into those vectors that do, and those that do not, generate maximal submodules.

Using singular vectors that generate maximal submodules, it is not too difficult to classify all possible degenerations of  $N=2$  Verma modules, since the structure of *submodules* is still relatively simple. However, it may become quite complicated to describe the same structure in terms of a restricted set of singular vectors that satisfy zero-twist annihilation conditions (and then, necessarily, in terms of subsingular vectors). The upshot is that, in the degenerate cases where several singular vectors exist in the module, their zero-twist representatives may lie in the section of the extremal diagram separated from the vectors

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<sup>2</sup>To avoid misunderstanding, let us point out explicitly that we do not claim, of course, that any maximal submodule in any  $N=2$  Verma module would be generated from one singular vector; this cannot be the case already for the sum of two submodules. What we are saying is that any maximal submodule is necessarily generated from an appropriate number of singular vectors that we work with in this paper. Note that this is not the case whenever subsingular vectors exist.

<sup>3</sup>In a context similar to that of the present paper, (diagrams of) extremal vectors were introduced in [FS]. Their usefulness in representation theory, which was pointed out in [FS], has been demonstrated in [ST, ST2, S2, FST].

generating the maximal submodule by a state that satisfies stronger highest-weight conditions and, thus, is a highest-weight vector in a twisted topological Verma submodule (diagrams (3.7), (3.21), and (3.43)). Depending on the relative positions of such topological singular vectors in the extremal diagram, therefore, a single picture in terms of extremal diagrams breaks into several cases in some of which the conventional singular vectors do, while in others do not, generate maximal submodules.

A careful analysis of the type of submodules in  $N = 2$  Verma modules is also crucial for correctly describing one particular degeneration of massive Verma modules where a massive Verma submodule is embedded into the direct sum of two twisted topological Verma submodules. In the conventional terms, the situation is described either as the existence of two linearly independent singular vectors with identical quantum numbers [D] or as the existence of a singular vector and a subsingular vector with identical quantum numbers (the latter case was missed in the conventional approach). In more invariant and in fact, much simpler terms, both these cases are described uniformly, as the existence of two singular vectors that satisfy twisted highest-weight conditions (diagram (3.42)). It follows that linearly independent singular vectors belong then to two twisted *topological* Verma submodules.

To summarize the situation with  $N = 2$  subsingular vectors, they are superfluous when it comes to classifying degenerations of  $N = 2$  Verma modules. Instead, our strategy is as follows. Given a submodule in the  $N = 2$  Verma module, we consider the entire extremal diagram that includes the vectors from which that submodule is generated. On the extremal diagram, then, we point out the states that generate the maximal submodule. These states, which turn out to satisfy twisted highest-weight conditions, are the singular vectors we work with in this paper. Restricting oneself to those extremal states (the conventional singular vectors) that fail to generate maximal submodules and classifying the ‘compensating’ subsingular vectors is then an exercise in describing *the same* structure of submodules in much less convenient terms. However, in order to make contact with the problems raised in the literature, we indicate in each of the degenerate cases<sup>4</sup> why and how the conventional singular vectors fail to generate maximal submodules; we then explicitly construct the corresponding subsingular vectors.

By choosing the singular vectors that satisfy twisted highest-weight conditions, we sacrifice the formal similarity with the case of Kač–Moody algebras, yet in the end of the day one observes [FST] that the structure of  $N = 2$  Verma modules is equivalent to the structure of certain modules over the affine  $\widehat{sl}(2)$  algebra: there is a functor from the category of ‘relaxed’  $\widehat{sl}(2)$  Verma modules introduced in [FST] to  $N = 2$  Verma modules. Restricted to the standard  $\widehat{sl}(2)$  Verma modules, this functor gives the twisted topological Verma modules over the  $N = 2$  algebra. The  $\widehat{sl}(2)$  singular vectors (which do generate maximal submodules) correspond then precisely to the  $N = 2$  singular vectors that we consider in this paper. This gives an intrinsic relation (actually, isomorphism [FST]) between affine  $\widehat{sl}(2)$  singular vectors and the  $N = 2$  singular vectors satisfying the twisted highest-weight conditions. Note also that the issue of subsingular vectors is normally not considered for the  $\widehat{sl}(2)$  algebra; combined with the equivalence proved in [FST], this clearly signifies, once again, that  $N = 2$  subsingular vectors are but an artifact of adopting the ‘zero-twist’ definition for singular vectors (as we explain in some detail after diagram (3.7)).

In what follows, we thus define and systematically refer to singular vectors that satisfy twisted ‘highest-weight’ conditions (see Definitions 2.3 and 2.9); the twist-zero singular vectors will be referred to as the conventional, ‘untwisted’, singular vectors (as we explain below, these can also be characterized as the top-level representatives of the extremal diagrams). As we have already mentioned, the two essentially

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<sup>4</sup>With one exception, where the classification of subsingular vectors would be too long in view of a large number of different cases of relative positions of the extremal diagrams describing the relevant submodules; classifying the subsingular vectors then remains a straightforward, although lengthy and unnecessary, exercise.

different types of  $N=2$  Verma modules are called the massive and the topological ones. ‘Highest-weight’ conditions will be used without quotation marks from now on. Whenever we talk about subsingular vectors, these will of course be understood in the setting where one restricts oneself to the conventional definition of singular vectors. ‘State’ is synonymous to ‘vector’. When applied to representations, the term ‘twisted’ means ‘transformed by the spectral flow’.

Our main results are the classification of degenerations of  $N=2$  Verma modules and the general construction of  $N=2$  singular vectors. We develop a systematic description of all possible degenerations of  $N=2$  Verma modules using the extremal diagrams. This allows us to describe the structure of submodules without invoking subsingular vectors. The formalism that we develop for the  $N=2$  algebra (see also [ST2]) is, at the same time, a natural counterpart of the construction of singular vectors of affine Lie algebras (see [MFF, FST]). To make contact with the issues discussed in the literature, we also show how the properties of the extremal diagrams determine whether or not the conventional singular vectors generate maximal submodules; when they do not, we give the general construction of the corresponding subsingular vectors that arise in the conventional approach.

In Section 2, we fix our notation and review the properties of the  $N=2$  algebra and singular vectors in its Verma modules. In Section 3, we describe all the degenerate cases where more than one singular vectors exist.

## 2 Preliminaries

### 2.1 The $N=2$ algebra, spectral flow transform, and Verma modules

The  $N=2$  superconformal algebra  $\mathcal{A}$  is taken in this paper in the following basis (see [ST2] for a discussion of the choice of various bases (and moddings) in the algebra):

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}, & [\mathcal{H}_m, \mathcal{H}_n] &= \frac{\mathbf{c}}{3}m\delta_{m+n,0}, \\ [\mathcal{L}_m, \mathcal{G}_n] &= (m-n)\mathcal{G}_{m+n}, & [\mathcal{H}_m, \mathcal{G}_n] &= \mathcal{G}_{m+n}, \\ [\mathcal{L}_m, \mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}, & [\mathcal{H}_m, \mathcal{Q}_n] &= -\mathcal{Q}_{m+n}, & m, n \in \mathbb{Z}. \\ [\mathcal{L}_m, \mathcal{H}_n] &= -n\mathcal{H}_{m+n} + \frac{\mathbf{c}}{6}(m^2 + m)\delta_{m+n,0}, \\ \{\mathcal{G}_m, \mathcal{Q}_n\} &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{\mathbf{c}}{3}(m^2 + m)\delta_{m+n,0}, \end{aligned} \quad (2.1)$$

The generators  $\mathcal{L}_m$ ,  $\mathcal{Q}_m$ ,  $\mathcal{H}_m$ , and  $\mathcal{G}_m$  are the Virasoro generators, the BRST current, the  $U(1)$  current, and the spin-2 fermionic current respectively.  $\mathcal{H}$  is not primary; instead, the commutation relations for the Virasoro generators are centreless. The element  $\mathbf{c}$  is central. Since it is diagonalizable in any representations (at least in all those that we are going to consider), we do not distinguish between  $\mathbf{c}$  and a number  $c \in \mathbb{C}$ , which it will be convenient to parametrize as

$$c = 3 \frac{t-2}{t} \quad (2.2)$$

with  $t \in \mathbb{C} \setminus \{0\}$ .

The spectral flow transform [SS, LVW] produces isomorphic images of the algebra  $\mathcal{A}$ . When applied to the generators of (2.1) it acts as

$$\mathcal{U}_\theta : \begin{aligned} \mathcal{L}_n &\mapsto \mathcal{L}_n + \theta\mathcal{H}_n + \frac{\mathbf{c}}{6}(\theta^2 + \theta)\delta_{n,0}, & \mathcal{H}_n &\mapsto \mathcal{H}_n + \frac{\mathbf{c}}{3}\theta\delta_{n,0}, \\ \mathcal{Q}_n &\mapsto \mathcal{Q}_{n-\theta}, & \mathcal{G}_n &\mapsto \mathcal{G}_{n+\theta}. \end{aligned} \quad (2.3)$$

For any  $\theta \in \mathbb{C}$ , this gives the mapping  $\mathcal{U}_\theta : \mathcal{A} \rightarrow \mathcal{A}_\theta$  of the  $N=2$  algebra  $\mathcal{A} \equiv \mathcal{A}_0$  to an isomorphic algebra  $\mathcal{A}_\theta$ , whose generators  $\mathcal{L}_n^\theta$ ,  $\mathcal{Q}_n^\theta$ ,  $\mathcal{H}_n^\theta$ , and  $\mathcal{G}_n^\theta$  (where  $\mathcal{X}_n^\theta = \mathcal{U}_\theta(\mathcal{X}_n)$ ) satisfy the same relations as those with  $\theta = 0$ . The family  $\mathcal{A}_\theta$  includes the Neveu–Schwarz and Ramond  $N=2$  algebras, as well as the algebras in which the fermion modes range over  $\pm\theta + \mathbb{Z}$ . Spectral flow is an *automorphism* when  $\theta \in \mathbb{Z}$ .

Next, consider Verma modules over the  $N=2$  algebra. As already mentioned in the Introduction, there are two different types of  $N=2$  Verma modules, the topological<sup>5</sup> and the massive ones. Since each of these can be ‘twisted’ by the spectral flow, we give the definitions of twisted modules, the ‘untwisted’ ones being recovered by setting the twist parameter  $\theta = 0$ . An important point, however, is that submodules of a given ‘untwisted’ module can be the twisted modules (which is the case with submodules in topological Verma modules and also with submodules determined by the ‘charged’ singular vectors).

**Definition 2.1** *A vector satisfying the highest-weight conditions<sup>6</sup>*

$$\begin{aligned} \mathcal{L}_m |h, t; \theta\rangle_{\text{top}} &= 0, \quad m \geq 1, & \mathcal{Q}_\lambda |h, t; \theta\rangle_{\text{top}} &= 0, \quad \lambda \in -\theta + \mathbb{N}_0 \\ \mathcal{H}_m |h, t; \theta\rangle_{\text{top}} &= 0, \quad m \geq 1, & \mathcal{G}_\nu |h, t; \theta\rangle_{\text{top}} &= 0, \quad \nu \in \theta + \mathbb{N}_0 \end{aligned} \quad \theta \in \mathbb{Z}, \quad (2.4)$$

with the Cartan generators having the following eigenvalues

$$\begin{aligned} (\mathcal{H}_0 + \tfrac{\varepsilon}{3}\theta) |h, t; \theta\rangle_{\text{top}} &= h |h, t; \theta\rangle_{\text{top}}, \\ (\mathcal{L}_0 + \theta\mathcal{H}_0 + \tfrac{\varepsilon}{6}(\theta^2 + \theta)) |h, t; \theta\rangle_{\text{top}} &= 0 \end{aligned} \quad (2.5)$$

is called the twisted topological highest-weight state. Conditions (2.4) are called the twisted topological highest-weight conditions.

**Definition 2.2** *The twisted topological Verma module  $\mathfrak{V}_{h,t;\theta}$  is freely generated from a twisted topological highest-weight state  $|h, t; \theta\rangle_{\text{top}}$  by*

$$\mathcal{L}_{-m}, \quad m \in \mathbb{N}, \quad \mathcal{H}_{-m}, \quad m \in \mathbb{N}, \quad \mathcal{Q}_{-m-\theta}, \quad m \in \mathbb{N}, \quad \mathcal{G}_{-m+\theta}, \quad m \in \mathbb{N}.$$

We write  $|h, t\rangle_{\text{top}} \equiv |h, t; 0\rangle_{\text{top}}$  and  $\mathcal{V}_{h,t} \equiv \mathfrak{V}_{h,t;0}$ .

$N=2$  modules are graded with respect to  $\mathcal{H}_0$  (the charge) and  $\mathcal{L}_0$  (the level). *Extremal vectors* in  $N=2$  modules are those having the minimal level for a fixed  $\mathcal{H}_0$ -charge. Associating a rectangular lattice with the bigrading, we have that increasing the  $\mathcal{H}_0$ -grade by 1 corresponds to shifting to the neighbouring site on the left, while increasing the level corresponds to moving down. The extremal vectors separate the lattice into those sites that are occupied by at least one element of the module and those that are not.

The extremal diagram of a topological Verma module reads (in the  $\theta = 0$  case for simplicity)

$$\begin{array}{c} |h, t\rangle_{\text{top}} \\ \swarrow \mathcal{G}_{-1} \quad \searrow \mathcal{Q}_{-1} \\ \bullet \quad \bullet \\ \swarrow \mathcal{G}_{-2} \quad \searrow \mathcal{Q}_{-2} \\ \bullet \quad \bullet \\ \vdots \quad \vdots \end{array} \quad (2.6)$$

<sup>5</sup>The name has to do with the fact that the highest-weight vectors in these modules correspond to primary states existing when the  $N=2$  algebra is viewed as the topological conformal algebra [EY, W].

<sup>6</sup>Here and henceforth,  $\mathbb{N} = 1, 2, \dots$ , while  $\mathbb{N}_0 = 0, 1, 2, \dots$

Then, *all* the states in the module are inside the ‘parabola’, while none of the states from the module are associated with the outside part of the plane. A characteristic feature of extremal diagrams of topological Verma modules is the existence of a state that satisfies stronger highest-weight conditions than the other states in the diagram. Geometrically, this is a ‘cusp’ point for the following reasons. Assigning grade  $-n$  to  $\mathcal{Q}_n$  and grade  $n$  to  $\mathcal{G}_n$ , we see that every two adjacent arrows in the diagram represent the operators whose grades differ by 1, except at the cusp, where they differ by 2. As we are going to see momentarily, the extremal diagrams of submodules in a (twisted) topological Verma module have ‘cusps’ as well, these ‘cusp’ points being the *topological singular vectors*:

**Definition 2.3** *A topological singular vector in the (twisted) topological Verma module  $\mathfrak{V}$  is any element of  $\mathfrak{V}$  that is not proportional to the highest-weight vector and satisfies twisted topological highest-weight conditions (i.e., is annihilated by the operators  $\mathcal{L}_m$ ,  $\mathcal{H}_m$ ,  $m \geq 1$ ,  $\mathcal{Q}_\lambda$ ,  $\lambda \in -\theta + \mathbb{N}_0$ , and  $\mathcal{G}_\nu$ ,  $\nu = \theta + \mathbb{N}_0$  with some  $\theta \in \mathbb{Z}$ ).*

The point is that the twist parameter  $\theta$  that enters the highest-weight conditions satisfied by the topological singular vector may be different from the twist parameter of the module. One readily shows, of course, that acting with the  $N = 2$  generators on a topological singular vector defined in this way generates a *submodule*.

The next statement follows from the results of [FST]:

**Theorem 2.4 ([FST])** *Any submodule of a (twisted) topological Verma module is generated from either one or two topological singular vectors.*

This is directly parallel to the situation encountered in affine  $\widehat{sl}(2)$  Verma modules — which, in fact, is the statement of [FST], where a functor was constructed that maps  $\widehat{sl}(2)$ -Verma modules to twisted topological Verma modules. The morphisms in a Verma modules category are singular vectors. The functor maps singular vectors in a  $\widehat{sl}(2)$ -Verma module to *topological* singular vectors and, thus, the assertion of the Theorem follows from the well known facts in the theory of  $\widehat{sl}(2)$ -Verma modules. Thus, a maximal submodule of a topological Verma module is either a twisted topological Verma module or a sum (*not* a direct one, of course) of two twisted topological Verma modules. In what follows, we call a submodule *primitive* if it is not a sum of two or more submodules.

Next, consider the massive  $N = 2$  Verma modules.

**Definition 2.5** *A twisted massive Verma module  $\mathfrak{U}_{h,\ell,t;\theta}$  is freely generated from a twisted massive highest-weight vector  $|h, \ell, t; \theta\rangle$  by the generators*

$$\mathcal{L}_{-m}, \quad m \in \mathbb{N}, \quad \mathcal{H}_{-m}, \quad m \in \mathbb{N}, \quad \mathcal{Q}_{-\theta-m}, \quad m \in \mathbb{N}_0, \quad \mathcal{G}_{\theta-m}, \quad m \in \mathbb{N}, \quad (2.7)$$

*The twisted massive highest-weight vector  $|h, \ell, t; \theta\rangle$  satisfies the following set of highest-weight conditions:*

$$\begin{aligned} \mathcal{Q}_{-\theta+m+1} |h, \ell, t; \theta\rangle &= \mathcal{G}_{\theta+m} |h, \ell, t; \theta\rangle = \mathcal{L}_{m+1} |h, \ell, t; \theta\rangle = \mathcal{H}_{m+1} |h, \ell, t; \theta\rangle = 0, \quad m \in \mathbb{N}_0, \\ (\mathcal{H}_0 + \frac{\epsilon}{3}\theta) |h, \ell, t; \theta\rangle &= h |h, \ell, t; \theta\rangle, \\ (\mathcal{L}_0 + \theta\mathcal{H}_0 + \frac{\epsilon}{6}(\theta^2 + \theta)) |h, \ell, t; \theta\rangle &= \ell |h, \ell, t; \theta\rangle. \end{aligned} \quad (2.8)$$

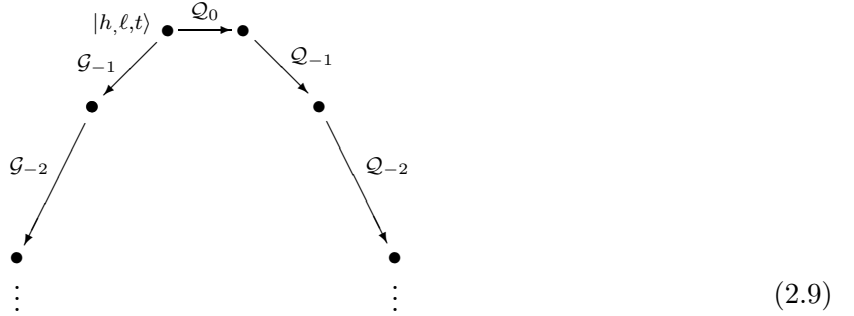
Equations (2.8) will be referred to as the *twisted massive highest-weight conditions*. It is understood that the twisted massive highest-weight vector does not satisfy the twisted *topological* highest-weight conditions

(i.e.,  $\mathcal{Q}_{-\theta}|h, \ell, t; \theta\rangle \neq 0$ ). The ordinary non-twisted case is obtained by setting  $\theta = 0$ . We identify  $|h, \ell, t\rangle \equiv |h, \ell, t; 0\rangle$  and  $\mathcal{U}_{h, \ell, t} \equiv \mathcal{U}_{h, \ell, t; 0}$ . When we say that a state in a Verma module satisfies (twisted) massive highest-weight conditions, we will mean primarily the annihilation conditions from (2.8).

An important property of the above definition is expressed by the following Lemma, which underlies all the subsequent analysis. The Lemma (which follows by a straightforward calculation in the universal enveloping algebra) is almost trivial, however we formulate it explicitly because of its wide use in what follows. Although we will not refer to the Lemma explicitly, the reader should keep in mind that it is implicit in almost all of our constructions.

**Lemma 2.6** *If a state  $|\theta'\rangle$  in a (twisted) massive Verma module satisfies the annihilation conditions (2.8) with the parameter  $\theta$  equal to  $\theta'$  then the states  $\mathcal{G}_{\theta'-N} \dots \mathcal{G}_{\theta'-1}|\theta'\rangle$ ,  $N \geq 1$ , and  $\mathcal{Q}_{-\theta'-N} \dots \mathcal{Q}_{-\theta'}|\theta'\rangle$ ,  $N \geq 0$ , satisfy annihilation conditions (2.8) with the parameter  $\theta$  equal to  $\theta' - N$  and  $\theta' + N + 1$  respectively.*

The states referred to in the Lemma fill out the extremal diagram of the massive Verma module. For  $\theta = 0$ , it reads as

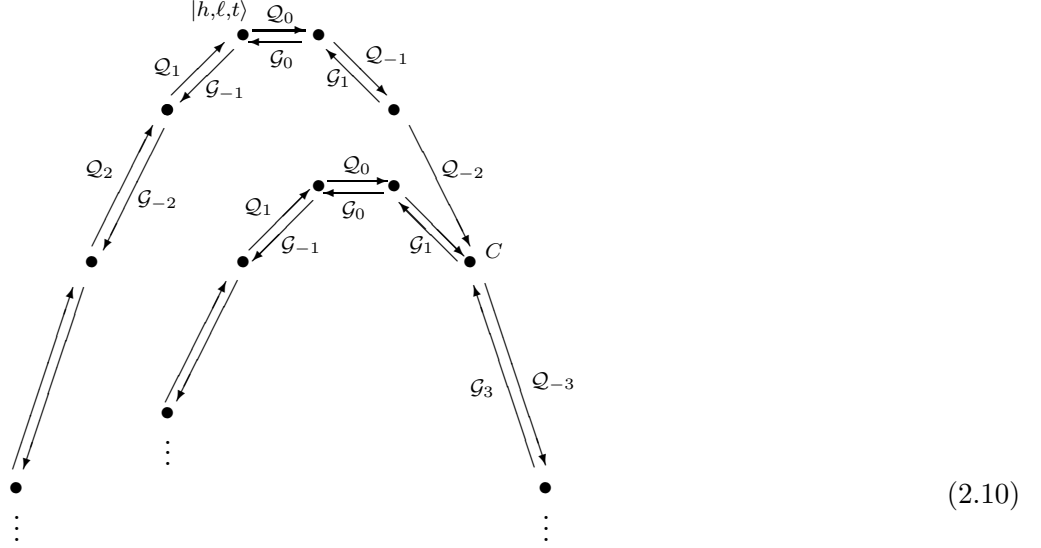


While  $|h, \ell, t\rangle$  satisfies the ‘untwisted’ highest-weight conditions ( $\theta = 0$  in (2.8)), the Lemma tells us that the other states in the extremal diagram satisfy twisted massive highest-weight conditions with all  $\theta \in \mathbb{Z}$ . The massive highest-weight vector must not be a ‘cusp point’ (i.e., it should not satisfy topological highest-weight conditions), however other cusp points may appear in the diagram depending on the highest-weight parameters  $(h, \ell, t)$ . Whenever this happens, there is a twisted topological submodule in the massive Verma module. The singular vectors appearing in the extremal diagrams of massive Verma modules are called “charged” for historical reasons [BFK].

**Definition 2.7** *The charged singular vector in a massive Verma module  $\mathcal{U}$  is any vector that satisfies twisted topological highest-weight conditions (2.4) (with whatever  $\theta \in \mathbb{Z}$ ) and belongs to the extremal diagram of the module.*

An example of a charged singular vector is given in the following diagram, where the twisted topological

highest-weight conditions (2.4) with  $\theta = 2$  are satisfied by the extremal state at the point  $C$ :



As a result, no operator inverts the action of  $Q_{-2}$ , while each of the other arrows is inverted *up to a scalar factor* by acting with the opposite mode of the other fermion. Thus, the extremal diagram branches at the ‘topological points’, and the crucial fact is that, once we are on the inner parabola, we can never leave it: none of the operators from the  $N = 2$  algebra map onto the remaining part of the big parabola from the small one, or, in other words, the inner diagram corresponds to an  $N = 2$  *submodule*. The general construction for the charged singular vectors is already obvious from the above remarks, and it will be given in Eqs. (2.40).

The ‘topological’ nature of submodules generated from charged singular vectors can be concealed if one allows submodules to be generated only from the conventional singular vectors, i.e. those that satisfy precisely the same highest-weight conditions as the highest-weight conditions satisfied by the highest-weight vector of the module. These conventional singular vectors do not in general coincide with the ‘cusp’ of the extremal diagram of the topological submodule. However, it is the existence of this ‘cusp’ that determines several crucial properties of the submodule.

Besides (twisted) topological Verma submodules, massive Verma modules may have submodules of the same, ‘massive’, type. These have to be clearly distinguished from the topological ones. The following definition will allow us to single out massive Verma modules.

**Definition 2.8** *Let  $|Y\rangle$  be a state in an  $N = 2$  Verma module that satisfies twisted massive highest-weight conditions with some  $\theta \in \mathbb{Z}$ . Then  $|X\rangle$  is said to be a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of  $|Y\rangle$  if either  $|X\rangle = \alpha \mathcal{G}_{\theta-N} \dots \mathcal{G}_{\theta-1} |Y\rangle$ ,  $N \in \mathbb{N}$ , or  $|X\rangle = \alpha \mathcal{Q}_{-\theta-M} \dots \mathcal{Q}_{-\theta} |Y\rangle$ ,  $M \in \mathbb{N}_0$ , where  $\alpha \in \mathbb{C}$ ,  $\alpha \neq 0$ .*

Those extremal states that do *not* generate the entire massive Verma module necessarily have a vanishing dense  $\mathcal{G}/\mathcal{Q}$ -descendant. In (2.10), for example, a part of the states on the extremal diagram generate only a submodule of the massive Verma module. Thus, in order to correctly define singular vectors that generate massive Verma submodules in a massive Verma module, one has to avoid vanishing dense  $\mathcal{G}/\mathcal{Q}$ -descendants of the singular vector. This is formalized in the following definition.

**Definition 2.9** *A representative of a massive singular vector in the massive Verma module  $\mathcal{U}_{h,\ell,t}$  is any element of  $\mathcal{U}_{h,\ell,t}$  such that*

- i) it is annihilated by the operators  $\mathcal{L}_m, \mathcal{H}_m, m \in \mathbb{N}$ ,  $\mathcal{Q}_\lambda, \lambda \in -\theta + \mathbb{N}$ , and  $\mathcal{G}_\nu, \nu = \theta + \mathbb{N}_0$  with some  $\theta \in \mathbb{Z}$ ,
- ii) none of its dense  $\mathcal{G}/\mathcal{Q}$ -descendants vanish,
- iii) the highest-weight state  $|h, \ell, t\rangle$  is not one of its descendants.

The meaning of the definition is that any representative of a massive singular vectors should generate an extremal diagram of the same type as the extremal diagram (2.9). On the other hand, vectors that do generate a given massive submodule can be chosen in different ways, and we thus talk about *representatives* of a massive singular vector.

In the conventional approach, the highest-weight conditions imposed on any singular vector read

$$\mathcal{Q}_{\geq 1} \approx \mathcal{G}_{\geq 0} \approx \mathcal{L}_{\geq 1} \approx \mathcal{H}_{\geq 1} \approx 0. \quad (2.11)$$

This selects the *top-level* (in accordance with the diagrams being drawn ‘upside-down’) representative of the extremal diagram of the submodule. These conventional, ‘untwisted’, singular vectors will thus be called top-level representatives. In [BFK, D], conditions (2.11) apply equally to the representatives of the charged and the massive singular vectors in our nomenclature. As regards the charged singular vectors, choosing the top-level representative conceals the fact that the submodule is of a different nature than the module itself; ignoring this then shows up in a number of ‘paradoxes’ when analyzing degenerations of the module.

In the general position, the massive singular vectors are equivalent to the ‘uncharged’ singular vectors in the conventional approach, since these generate the same submodule. In the degenerate cases, however, the conventional, top-level, singular vectors may not generate the entire submodule generated from some other states on the same extremal diagram. This depends on the properties of the extremal diagram of the submodule, which change when there appears a charged singular vector, i.e., when one of the extremal states in the diagram happens to satisfy twisted *topological* highest-weight conditions. The conventional representatives of singular vectors may then be separated by such topological points from those sections of the extremal diagram which generate the maximal submodule.

Our strategy is to define and explicitly construct singular vectors that lie in the ‘safe’ sections of the extremal diagrams (those from which maximal submodules are generated). As we have mentioned, this eliminates the notion of subsingular vectors. Describing the structure of  $N=2$  modules in this way appears to be more transparent and in any case much more economical, considering a fast proliferation of cases describing the subsingular vectors that have to be introduced whenever the conventional, top-level, singular vectors lie in the ‘wrong’ section of the extremal diagram of the submodule. However, given the analysis that follows, it is a straightforward exercise to classify all such cases (and explicitly construct the subsingular vectors) by looking at how the extremal diagram is divided into different sections by the topological singular vectors.

In the next subsection, we develop the algebraic formalism that allows us to construct singular vectors. The reader who is interested only in the degeneration patterns may skip to Subsection 2.3 and Section 3.

## 2.2 The algebra of continued operators

In order to explicitly construct singular vectors, we follow ref. [ST2] in making use of ‘continued’ operators that generalize the dense  $\mathcal{G}/\mathcal{Q}$ -descendants to the case of non-integral (in fact, complex)  $\theta$ .

The new operators  $g(a, b)$  and  $q(a, b)$  can be thought of as a continuation of the products of modes  $\mathcal{G}_a \mathcal{G}_{a+1} \dots \mathcal{G}_{a+N}$  and  $\mathcal{Q}_a \mathcal{Q}_{a+1} \dots \mathcal{Q}_{a+N}$ , respectively, to a complex number of factors. In particular, whenever the *length*  $b - a + 1$  of  $g(a, b)$  or  $q(a, b)$  is a non-negative integer, the corresponding operator becomes, by definition, the product of the corresponding modes:

$$g(a, b) = \prod_{i=0}^{L-1} \mathcal{G}_{a+i}, \quad q(a, b) = \prod_{i=0}^{L-1} \mathcal{Q}_{a+i}, \quad \text{iff } L \equiv b - a + 1 = 0, 1, 2, \dots \quad (2.12)$$

(in the case where  $L = 0$ , the product evaluates as 1).

We now postulate a number of properties of the new operators in such a way that these properties become identities whenever the operators reduce to elements of the universal enveloping algebra. This is analogous to the well-known story about complex exponents in the construction of [MFF].

To begin with, the idea of a ‘dense’ filling with fermions is formalized in the rules

$$\begin{aligned} g(a, b-1) g(b, \theta-1) |\theta\rangle_g &= g(a, \theta-1) |\theta\rangle_g, \\ q(a, b-1) q(b, -\theta-1) |\theta\rangle_q &= q(a, -\theta-1) |\theta\rangle_q, \end{aligned} \quad a, b, \theta \in \mathbb{C}, \quad (2.13)$$

where  $|\theta\rangle_g$  is any state that satisfies  $\mathcal{G}_{\theta+n} |\theta\rangle_g = 0$  for  $n \in \mathbb{N}_0$ , and  $|\theta\rangle_q$ , similarly, satisfies  $\mathcal{Q}_{-\theta+n} |\theta\rangle_q = 0$  for  $n \in \mathbb{N}_0$ .

Under the spectral flow transform (2.3), the operators  $g(a, b)$  and  $q(a, b)$  behave in the manner that is also inherited from the behaviour of the products (2.12):

$$\mathcal{U}_\theta : \begin{aligned} g(a, b) &\mapsto g(a + \theta, b + \theta), \\ q(a, b) &\mapsto q(a - \theta, b - \theta). \end{aligned} \quad (2.14)$$

Further properties of the new operators originate in the fact that, the  $N=2$  generators  $\mathcal{Q}$  and  $\mathcal{G}$  being fermions, they satisfy the vanishing formulae such as, e.g.,  $\mathcal{G}_n \cdot \prod_{i=a}^{a+N} \mathcal{G}_i = 0$ ,  $N \in \mathbb{N}_0$ ,  $a \leq n \leq a + N$ . For complex values of the parameters, we impose

$$\mathcal{G}_a g(b, c) = 0, \quad \mathcal{Q}_a q(b, c) = 0, \quad a - b \in \mathbb{N}_0 \quad \text{and} \quad (a - c \notin \mathbb{N} \quad \text{or} \quad b - c - 1 \in \mathbb{N}). \quad (2.15)$$

Similarly, the ‘left-hand’ annihilation properties are expressed by the relations

$$g(a, b) \mathcal{G}_c = 0, \quad q(a, b) \mathcal{Q}_c = 0, \quad b - c \in \mathbb{N}_0 \quad \text{and} \quad (a - c \notin \mathbb{N} \quad \text{or} \quad a - b - 1 \in \mathbb{N}). \quad (2.16)$$

The formulae to commute the continued operators with the bosons  $\mathcal{L}_{\geq 1}$  and  $\mathcal{H}_{\geq 1}$  read

$$\begin{aligned} [\mathcal{K}_p, g(a, b)] &= \sum_{l=0}^{d(p, a, b)} g(a, b - l - 1) [\mathcal{K}_p, \mathcal{G}_{b-l}] \mathcal{G}_{b-l+1} \dots \mathcal{G}_b, \quad p \in \mathbb{N}, \\ [\mathcal{K}_p, q(a, b)] &= \sum_{l=0}^{d(p, a, b)} q(a, b - l - 1) [\mathcal{K}_p, \mathcal{Q}_{b-l}] \mathcal{Q}_{b-l+1} \dots \mathcal{Q}_b, \end{aligned} \quad (2.17)$$

where  $\mathcal{K} = \mathcal{L}$  or  $\mathcal{H}$ , and

$$d(p, a, b) = \begin{cases} b - a, & p - b + a \in \mathbb{N}_0 \quad \text{and} \quad b - a + 1 \in \mathbb{N}_0, \\ p - 1, & \text{otherwise.} \end{cases} \quad (2.18)$$

The main point here is that, even though the length  $b - a + 1$  may not be an integer, there is always an integral number of terms on the RHS of (2.17).

Similarly, applying the  $g$  and  $q$  operators changes the eigenvalues of  $\mathcal{L}_0$  and  $\mathcal{H}_0$ , which can be expressed by the commutation relations

$$\begin{aligned} [\mathcal{L}_0, g(a, b)] &= -\frac{1}{2}(a+b)(b-a+1)g(a, b), & [\mathcal{H}_0, g(a, b)] &= (b-a+1)g(a, b), \\ [\mathcal{L}_0, q(a, b)] &= -\frac{1}{2}(a+b)(b-a+1)q(a, b), & [\mathcal{H}_0, q(a, b)] &= (-b+a-1)q(a, b). \end{aligned} \quad (2.19)$$

Further annihilation properties with respect to the fermionic operators are as follows:

$$\mathcal{Q}_{-\theta+n} g(\theta, -1) |h, \ell, t\rangle = 0, \quad \theta \in \mathbb{C}, \quad n \in \mathbb{N}, \quad (2.20)$$

while

$$\mathcal{Q}_{-\theta} g(\theta, -1) |h, \ell, t\rangle = 2(\ell + \theta h - \frac{1}{t}(\theta^2 + \theta)) g(\theta + 1, -1) |h, \ell, t\rangle. \quad (2.21)$$

It is understood here that the operators acting from the left of  $g(\theta, \theta' - 1)$  or  $q(-\theta, -\theta' - 1)$  are the  $N=2$  generators from (2.1) subjected to the spectral flow transform  $\mathcal{U}_\theta$ . Similarly, for the  $q$ -operators, we have the following properties:

$$\begin{aligned} \mathcal{G}_{\theta+n} q(-\theta, 0) |h, \ell, t\rangle &= 0, \quad n \in \mathbb{N}, \\ \mathcal{G}_\theta q(-\theta, 0) |h, \ell, t\rangle &= 2(\ell + \theta h - \frac{1}{t}(\theta^2 + \theta)) q(-\theta + 1, 0) |h, \ell, t\rangle. \end{aligned} \quad (2.22)$$

For the ‘continued’ *topological* highest-weight states we have, in the same manner,

$$\mathcal{Q}_{-\theta'} g(\theta', \theta - 1) |h, t; \theta\rangle_{\text{top}} = 2(\theta' - \theta)(h + \frac{1}{t}(\theta - \theta' - 1)) g(\theta' + 1, \theta - 1) |h, t; \theta\rangle_{\text{top}} \quad (2.23)$$

$$\mathcal{G}_{\theta'} q(-\theta', -\theta - 1) |h, t; \theta\rangle_{\text{top}} = 2(\theta' - \theta)(h + 1 + \frac{1}{t}(\theta - \theta' - 1)) q(\theta' + 1, \theta - 1) |h, t; \theta\rangle_{\text{top}} \quad (2.24)$$

The formulae to commute the negative-moded  $\mathcal{H}$  and  $\mathcal{L}$  operators through  $q(a, b)$  and  $g(a, b)$  read

$$\begin{aligned} [g(a, b), \mathcal{K}_p] &= \sum_{l=0}^{d(-p, a, b)} \mathcal{G}_a \dots \mathcal{G}_{a+l-1} [\mathcal{G}_{a+l}, \mathcal{K}_p] g(a+l+1, b), \\ [q(a, b), \mathcal{K}_p] &= \sum_{l=0}^{d(-p, a, b)} \mathcal{Q}_a \dots \mathcal{Q}_{a+l-1} [\mathcal{Q}_{a+l}, \mathcal{K}_p] q(a+l+1, b), \end{aligned} \quad (2.25)$$

where  $d(p, a, b)$  is given by (2.18). As before,  $\mathcal{K} = \mathcal{H}$  or  $\mathcal{L}$ .

The formulae postulated for  $g$  and  $q$  make up a consistent set of algebraic rules (in particular, they are consistent with operator associativity and with the positive integral length reduction (2.12)). All the properties listed above make it easy to show the following

**Lemma 2.10**

I. A massive highest-weight state maps under the action of operators  $g$  and  $q$  into the states  $g(\theta, -1)|h, \ell, t\rangle$  and  $q(-\theta, 0)|h, \ell, t\rangle$  that satisfy the following annihilation conditions:

$$\begin{aligned} \mathcal{L}_m g(\theta, -1) |h, \ell, t\rangle &= 0, \quad m \in \mathbb{N}, \\ \mathcal{H}_m g(\theta, -1) |h, \ell, t\rangle &= 0, \quad m \in \mathbb{N}, \\ \mathcal{G}_a g(\theta, -1) |h, \ell, t\rangle &= 0, \quad a \in \theta + \mathbb{N}_0, \\ \mathcal{Q}_a g(\theta, -1) |h, \ell, t\rangle &= 0, \quad a \in -\theta + \mathbb{N}, \end{aligned} \quad (2.26)$$

and

$$\begin{aligned}
\mathcal{L}_m q(-\theta, 0) |h, \ell, t\rangle &= 0, \quad m \in \mathbb{N}, \\
\mathcal{H}_m q(-\theta, 0) |h, \ell, t\rangle &= 0, \quad m \in \mathbb{N}, \\
\mathcal{G}_a q(-\theta, 0) |h, \ell, t\rangle &= 0, \quad a \in \theta + \mathbb{N}, \\
\mathcal{Q}_a q(-\theta, 0) |h, \ell, t\rangle &= 0, \quad a \in -\theta + \mathbb{N}_0.
\end{aligned} \tag{2.27}$$

II. *The twisted topological highest-weight states are mapped under the action of  $g$  and  $q$  into the states that satisfy*

$$\begin{aligned}
\mathcal{L}_m g(\theta', \theta - 1) |h, t; \theta\rangle_{\text{top}} &= 0, \quad \mathcal{L}_m q(\theta', -\theta - 1) |h, t; \theta\rangle_{\text{top}} = 0, \quad m \in \mathbb{N}. \\
\mathcal{H}_m g(\theta', \theta - 1) |h, t; \theta\rangle_{\text{top}} &= 0, \quad \mathcal{H}_m q(\theta', -\theta - 1) |h, t; \theta\rangle_{\text{top}} = 0, \quad m \in \mathbb{N}, \\
\mathcal{G}_a g(\theta', \theta - 1) |h, t; \theta\rangle_{\text{top}} &= 0, \quad \mathcal{G}_a q(\theta', -\theta - 1) |h, t; \theta\rangle_{\text{top}} = 0, \quad a \in -\theta' + \mathbb{N}. \\
\mathcal{Q}_a g(\theta', \theta - 1) |h, t; \theta\rangle_{\text{top}} &= 0, \quad \mathcal{Q}_a q(\theta', -\theta - 1) |h, t; \theta\rangle_{\text{top}} = 0, \quad a \in \theta' + \mathbb{N}_0,
\end{aligned} \tag{2.28}$$

These equations allow us to relate the states satisfying the highest-weight conditions with different twists, which is necessary for the construction of singular vectors.

It is also useful to know the parameters (the corresponding  $h$  and  $\ell$ ) of the vector obtained from  $|h, \ell, t; \theta\rangle$  by the action of a  $q$ - or a  $g$ -operator. These are described as follows: up to a numerical coefficient, we have

$$\begin{aligned}
g(\theta', \theta - 1) |h, \ell, t; \theta\rangle &\sim |h', \ell', t; \theta'\rangle, \\
h' &= h + \frac{2}{t}(\theta - \theta'), \\
\ell' &= \ell + (\theta' - \theta)(h - \frac{1}{t}(\theta' - \theta + 1)),
\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}
q(-\theta', -\theta) |h, \ell, t; \theta\rangle &\sim |h'', \ell'', t; \theta' + 1\rangle, \\
h'' &= h + \frac{2}{t}(\theta - \theta' - 1), \\
\ell'' &= \ell + (\theta' - \theta + 1)(h - \frac{1}{t}(\theta' - \theta + 2)).
\end{aligned} \tag{2.30}$$

Note that whenever  $\ell + (\theta' - \theta)h - \frac{1}{t}((\theta' - \theta)^2 + \theta' - \theta) = 0$ , Eqs. (2.22) allow us to show that, in addition to (2.30),

$$q(-\theta', -\theta) |h, \ell, t; \theta\rangle \sim \left| h + \frac{2}{t}(\theta - \theta') - 1, t; \theta' \right\rangle_{\text{top}}. \tag{2.31}$$

In what follows, the above formulae will be used to construct the general expressions for singular vectors in  $N=2$  Verma modules.

### 2.3 Singular vectors in codimension 1

In the general position, there are no singular vectors in Verma modules. Singular vectors can appear in codimension 1, when there is 1 relation between parameters of the highest-weight state. This is considered in the present subsection, while the cases of a higher codimension, where several singular vectors coexist in the module, are considered in the next section. We begin with the topological Verma modules. As we are going to see, this case is also crucial for the massive Verma modules, since the analysis of the latter reduces, to a considerable degree, to the analysis of certain topological Verma modules.

**Theorem 2.11**

I. A singular vector exists in the topological Verma module  $\mathcal{V}_{h,t}$  if and only if  $h = \mathbf{h}^+(r, s, t)$  or  $h = \mathbf{h}^-(r, s, t)$ , where

$$\begin{aligned}\mathbf{h}^+(r, s, t) &= -\frac{r-1}{t} + s - 1, \\ \mathbf{h}^-(r, s, t) &= \frac{r+1}{t} - s,\end{aligned}\quad r, s \in \mathbb{N}.$$
(2.32)

II. All singular vectors in the topological Verma module  $\mathcal{V}_{\mathbf{h}^\pm(r,s,t),t}$  over the  $N=2$  superconformal algebra are given by the explicit construction:

$$\begin{aligned}|E(r, s, t)\rangle^+ &= g(-r, (s-1)t-1) q(-(s-1)t, r-1-t) \dots g((s-2)t-r, t-1) q(-t, r-1-t(s-1)) \\ &\quad \cdot g((s-1)t-r, -1) |\mathbf{h}^+(r, s, t), t\rangle_{\text{top}},\end{aligned}$$
(2.33)

$$\begin{aligned}|E(r, s, t)\rangle^- &= q(-r, (s-1)t-1) g(-(s-1)t, r-t-1) \dots q((s-2)t-r, t-1) g(-t, r-1-(s-1)t) \\ &\quad \cdot q((s-1)t-r, -1) |\mathbf{h}^-(r, s, t), t\rangle_{\text{top}},\end{aligned}$$
(2.34)

where  $r, s \in \mathbb{N}$  and the factors in the first line of each formula are

$$g(-r-t-mt+st, -1+mt) q(-mt, r-1+mt-st), \quad s-1 \geq m \geq 1$$
(2.35)

and

$$q(-r-t-mt+st, -1+mt) g(-mt, r-1+mt-st), \quad s-1 \geq m \geq 1,$$
(2.36)

respectively. The  $|E(r, s, t)\rangle^\pm$  singular vectors satisfy twisted topological highest-weight conditions with the twist parameter  $\theta = \mp r$ , are on the level  $rs + \frac{1}{2}r(r-1)$  over the corresponding topological highest-weight state, and have the relative charge  $\pm r$ .

In what follows, we will need singular vector operators  $\mathcal{E}^\pm(r, s, t)$  such that

$$|E(r, s, t)\rangle^\pm = \mathcal{E}^\pm(r, s, t) |\mathbf{h}^\pm(r, s, t), t\rangle_{\text{top}}.$$

In a direct analogy with the well-known affine Lie algebra case [MFF, Ma], “all singular vectors” applies literally to non-rational  $t$ , while for rational  $t$ , a singular vector may be given already by a subformula of Eqs. (2.33), (2.34) as soon as that subformula (obtained by dropping several  $g$ - and  $q$ -operators from the left) produces an element of the Verma module.

To avoid a possible misunderstanding, let us point out once again that a given submodule may be generated from a state other than the singular vectors we work with (in the present case, other than the topological singular vectors). This is completely similar to the situation in the standard case of (affine) Lie algebras, where it is possible to generate a given Verma submodule from some vectors other than the highest-weight state of the submodule. However, any such vector is a descendant of the highest-weight vector and, in this sense, considering it as a ‘singular vector’ is unnecessary (and, often, inconvenient). An essential point about singular vectors (2.33), (2.34) is that the corresponding submodules can be *freely* generated from these vectors.

PROOF. Part I was conjectured in [S1] and proved in [FST] as an immediate consequence of the equivalence result obtained there. The construction of singular vectors in Part II is borrowed from [ST], while the fact that these are all singular vectors follows again from [FST]. The scheme to evaluate the singular vectors as elements of the topological Verma module can be outlined as follows. Consider, for definiteness, (2.34). This can be rewritten as

$$|E(r, s, t)\rangle^- = q(-r, (s-1)t-1) \mathcal{E}^{+, r-(s-1)t}(r, s-1, t) q((s-1)t-r, -1) |\mathbf{h}^-(r, s, t), t\rangle_{\text{top}},$$

where  $\mathcal{E}^{+, \theta}(r, s-1, t)$  is the spectral flow transform of the singular vector operator. Now, assuming that this operator is already expressed in terms of modes of  $\mathcal{L}$ ,  $\mathcal{H}$ ,  $\mathcal{G}$ , and  $\mathcal{Q}$ , we shall prove that  $|E(r, s, t)\rangle^-$  is a polynomial in  $\mathcal{L}_{\leq -1}$ ,  $\mathcal{H}_{\leq -1}$ ,  $\mathcal{G}_{\leq -1}$ , and  $\mathcal{Q}_{\leq -1}$  acting on  $|\mathbf{h}^-(r, s, t), t\rangle_{\text{top}}$ . To this end, we use (2.13) to rewrite  $q(-r, (s-1)t-1)\mathcal{E}^{+, r-(s-1)t}(r, s-1, t)$  as  $q(-r, (s-1)t-r-1)\mathcal{Q}_{(s-1)t-r}\dots\mathcal{Q}_{(s-1)t-1}\mathcal{E}^{+, r-(s-1)t}(r, s-1, t)$  and observe that all of the operators  $\mathcal{Q}_{(s-1)t-r}, \dots, \mathcal{Q}_{(s-1)t-1}$  annihilate the state  $q((s-1)t-r, -1) \cdot |\mathbf{h}^-(r, s, t), t\rangle_{\text{top}}$  in accordance with (2.20)–(2.24). After commuting these operators to the right, Eqs. (2.16) apply to  $q(-r, (s-1)t-r-1)$  and all of the remaining modes of  $\mathcal{Q}$ . Finally, we are left with a polynomial in the modes of only  $\mathcal{L}$  and  $\mathcal{H}$  between  $q(-r, (s-1)t-r-1)$  and  $q((s-1)t-r, -1)$ . Then, using Eqs. (2.25), we see that the two  $q$ -operators meet each other and are eliminated using Eqs. (2.13) and (2.12). Thus, we are left with a polynomial in  $\mathcal{L}_{\leq -1}$ ,  $\mathcal{H}_{\leq -1}$ ,  $\mathcal{G}_{\leq -1}$ , and  $\mathcal{Q}_{\leq -1}$  acting on  $|\mathbf{h}^-(r, s, t), t\rangle_{\text{top}}$ . This allows us to develop the induction argument, with the starting point being that in the *center* of each of the formulas (2.33) and (2.34), there is a  $g$ - or  $q$ -operator of the positive integral length  $r$ , which therefore reduces to the product of modes according to (2.12).  $\square$

We now turn to singular vectors in massive  $N=2$  Verma modules. To a given massive Verma module  $\mathcal{U}_{h, \ell, t}$  we associate four twisted topological Verma modules whose highest-weight vectors are the “continued” states of the form of those entering (2.26) and (2.27). Namely, let  $\theta'$  and  $\theta'' = -\theta' + ht - 1$  be two roots of the equation

$$\ell = -\theta h + \frac{1}{t}(\theta^2 + \theta). \quad (2.37)$$

Then, using Lemma 2.10 and Eqs. (2.29) and (2.30), it is immediately verified that the states

$$\begin{aligned} g(\theta', -1)|h, \ell, t\rangle, & \quad q(-\theta', 0)|h, \ell, t\rangle, \\ g(\theta'', -1)|h, \ell, t\rangle, & \quad q(-\theta'', 0)|h, \ell, t\rangle, \end{aligned} \quad (2.38)$$

formally satisfy the twisted topological highest-weight conditions (2.4), although possibly with a complex twist parameter.

We will say, for brevity, that a highest-weight state admits a singular vector if the corresponding singular vector exists in the module built on that state and that a highest-weight state admits no singular vectors if no singular vectors exist in the module. As it turns out, all possible degenerations of the massive Verma module  $\mathcal{U}_{h, \ell, t}$  occur depending on whether and how many of states (2.38) belong to  $\mathcal{U}_{h, \ell, t}$  and/or admit a topological singular vector. We now introduce a stratification of the space of highest weights  $(h, \ell, t)$  controlled by the behaviour of vectors (2.38). In the subsequent sections, we consider each stratum in turn and study the corresponding degenerations of massive Verma modules. The possible cases, whose labels  $\mathcal{O}_{\text{xyz}}$  indicate the existence of typical (massive or charged) singular vectors, are as follows:

**codimension 0:**

1. none of states (2.38) belong to  $\mathcal{U}_{h, \ell, t}$  and at least one of states (2.38) admits no topological singular vectors;

**codimension 1:**

2.  $\mathcal{O}_{\text{m}}$ : one of the states (2.38) admits precisely one topological singular vector, each of the other states (2.38) admits at least one topological singular vector, while none of states (2.38) belong to  $\mathcal{U}_{h, \ell, t}$ ;
3.  $\mathcal{O}_{\text{c}}$ : one and only one of states (2.38) belongs to  $\mathcal{U}_{h, \ell, t}$  and none of states (2.38) admit a topological singular vector;

codimension 2:

4.  $\mathcal{O}_{\text{mm}}$ : each of states (2.38) admits at least two distinct topological singular vectors, while none of states (2.38) belong to  $\mathcal{U}_{h,\ell,t}$ ;
5.  $\mathcal{O}_{\text{cc}}$ : precisely one of the states from each column in (2.38) belongs to the module  $\mathcal{U}_{h,\ell,t}$  and none of these two states admit a topological singular vector;
6.  $\mathcal{O}_{\text{cm}}$ : one of the states from (2.38) belongs to the module  $\mathcal{U}_{h,\ell,t}$  and admits precisely one topological singular vector; none of states (2.38) admit two different topological singular vectors; no two states from different columns in (2.38) belong to  $\mathcal{U}_{h,\ell,t}$ ;

codimension 3:

7.  $\mathcal{O}_{\text{cmm}}$ : one of the states from (2.38) belongs to the module  $\mathcal{U}_{h,\ell,t}$  and admits at least two different topological singular vectors; no two states from different columns in (2.38) belong to  $\mathcal{U}_{h,\ell,t}$ ;
8.  $\mathcal{O}_{\text{ccm}}$ : precisely one of the states from each column in (2.38) belongs to the module  $\mathcal{U}_{h,\ell,t}$ ; each of these two states admits at least one topological singular vector.

In what follows, we will often refer to cases 1–8 by saying that the highest-weight parameters  $(h, \ell, t)$  of  $\mathcal{U}_{h,\ell,t}$  belong to the corresponding  $\mathcal{O}_{\text{xyz}}$ .

**Lemma 2.12** *The above cases 1–8 divide the space of highest-weight parameters  $(h, \ell, t)$  into a disjoint union.*

PROOF. Observe, first of all, that each case in the above list is singled out by a combination of two conditions or their negations: that one of states (2.38) belongs to the module  $\mathcal{U}_{h,\ell,t}$  and that a (necessarily topological) singular vector exists in the module built on one of states (2.38). The first condition means that the  $\theta$  parameter is an integer of the appropriate sign such that formulae (2.12) apply and, thus, the corresponding state in an element of  $\mathcal{U}_{h,\ell,t}$ . We find from (2.37) that the condition for this to be the case is  $\ell = n(h + \frac{n-1}{t})$ ,  $n \in \mathbb{Z}$ . Next, whether or not a state from (2.38) admits a topological singular vector is a matter of whether the corresponding  $h'$  or  $h''$  parameter determined according to (2.29) and (2.30) equals one of the  $\mathfrak{h}^\pm$  from (2.32). We see from (2.37) that this is the case if and only if  $\ell = -\frac{t}{4}(h - \mathfrak{h}^-(r, s, t))(h - \mathfrak{h}^+(r, s+1, t))$ ,  $r, s \in \mathbb{N}$ . Note that the two expressions for  $\ell$  are precisely the zeros of the Kač determinant [BFK].

The cases 1–8 do not overlap by construction; on the other hand, there are no other possible combinations of the two basic conditions, since such combinations (e.g., that *three distinct* states from (2.38) belong to  $\mathcal{U}_{h,\ell,t}$ , etc.) would either lead to an overdetermined system of equations on the parameters  $h, t, \theta'$ , and  $\theta''$ , which admits no solutions, or would contradict the embedding patterns of topological Verma modules, which are isomorphic [FST] to the embedding patterns of  $\widehat{\mathfrak{sl}}(2)$  Verma modules.  $\square$

Unless one considers cases of codimension 2 or 3, there is no discrepancy in the use of the term ‘charged’ between the present paper and the treatment of [BFK] (and similarly with the correspondence ‘massive’–‘uncharged’), in the sense that the top-level representatives of singular vectors generate exactly the same submodules as our singular vectors. In the following two Theorems, we take care not to slip down to a higher codimension and recover the ‘charged’ and the ‘massive’ cases:

**Theorem 2.13**

I. *The highest-weight of the massive Verma module  $\mathcal{U}_{h,\ell,t}$  belongs to the set  $\mathcal{O}_c$  if and only if  $\ell = l_{\text{ch}}(n, h, t)$ , where*

$$l_{\text{ch}}(n, h, t) = n(h + \frac{n-1}{t}), \tag{2.39}$$

$$(n, h, t) \in (\mathbb{Z} \times \mathbb{C} \times \mathbb{C}) \setminus \left\{ (n', s - \frac{2n'-1+r}{t'}, t') \mid r, s \in \mathbb{Z}, r \neq 0, r \cdot s \geq 0, n' \in \mathbb{Z}, t' \in \mathbb{C} \right\}.$$

II. Then, the massive Verma module  $\mathcal{U}_{h, l_{\text{ch}}(n, h, t), t}$  contains precisely one submodule, which is generated from the charged singular vector

$$|E(n, h, t)\rangle_{\text{ch}} = \begin{cases} \mathcal{Q}_n \dots \mathcal{Q}_0 |h, l_{\text{ch}}(n, h, t), t\rangle & n \leq 0, \\ \mathcal{G}_{-n} \dots \mathcal{G}_{-1} |h, l_{\text{ch}}(n, h, t), t\rangle, & n \geq 1. \end{cases} \quad (2.40)$$

Every such vector satisfies the twisted topological highest-weight conditions (2.4) with  $\theta = -n$  and, therefore, the submodule is isomorphic to a twisted topological Verma module.

PROOF. The formula for  $l_{\text{ch}}(n, h, t)$  is obvious from the proof of Lemma 2.12; the condition  $\ell = l_{\text{ch}}(n, h, t)$  is equivalent to the fact that a solution of Eq. (2.37) is an integer ( $\theta' \in \mathbb{Z}$  or  $\theta'' \in \mathbb{Z}$ ). This reproduces the ‘charged’ series of zeros of the Kač determinant [BFK]. The excluded set  $\mathbb{X}(n, t)$  is that where other zeros of the Kač determinant occur. Finally, a straightforward calculation in the universal enveloping algebra shows that the state (2.40) does satisfy the twisted topological highest-weight conditions, which completes the proof.  $\square$

The top-level representative of (2.40), which reads as

$$|s(n, h, t)\rangle_{\text{ch}} = \begin{cases} \mathcal{G}_0 \dots \mathcal{G}_{-n-1} |E(n, h, t)\rangle_{\text{ch}}, & n \leq 0, \\ \mathcal{Q}_1 \dots \mathcal{Q}_{n-1} |E(n, h, t)\rangle_{\text{ch}}, & n \geq 1, \end{cases} \quad (2.41)$$

is the conventional charged singular vector satisfying the conditions given in [BFK]. Thus, the conventional charged singular vector necessarily belongs to a twisted topological Verma submodule, and it is the highest-weight vector of this submodule that we call the charged singular vector  $|E(n, h, t)\rangle_{\text{ch}}$  in this paper.

Further, as regards the massive singular vectors, we have

### Theorem 2.14

I. The highest-weight of the massive Verma module  $\mathcal{U}_{h, \ell, t}$  belongs to the set  $\mathcal{O}_{\text{m}}$  if and only if  $\ell = l(r, s, h, t)$ , where

$$\begin{aligned} l(r, s, h, t) &= -\frac{t}{4}(h - h^-(r, s, t))(h - h^+(r, s + 1, t)), \\ (r, s, h, t) &\in \left( \mathbb{N} \times \mathbb{N} \times \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Q}) \cup \mathbb{Y} \right) \setminus \left\{ (r', s', \pm s' - \frac{2n-1 \pm r'}{t'}, t') \mid n \in \mathbb{Z}, r', s' \in \mathbb{N}, t' \in \mathbb{C} \right\}, \end{aligned} \quad (2.42)$$

where

$$\begin{aligned} \mathbb{Y} &= \left\{ (r', s', h', -\frac{p}{q}) \mid 1 \leq r' \leq p, 1 \leq s' \leq q, p, q \in \mathbb{N}, h' \in \mathbb{C} \right\} \cup \\ &\quad \left\{ (r', 1, h', -\frac{p}{q}) \mid p+1 \leq r' \leq 2p, p, q \in \mathbb{N}, h' \in \mathbb{C} \right\} \end{aligned} \quad (2.43)$$

In this case,  $\mathcal{U}_{h, \ell, t}$  contains precisely one submodule, which is a massive Verma module.

II. Then, the representatives of the massive singular vector in the massive Verma module  $\mathcal{U}_{h, l(r, s, h, t), t}$  are given by

$$|S(r, s, h, t)\rangle^- = g(-rs, r + \theta^-(r, s, h, t) - 1) \mathcal{E}^{-, \theta^-(r, s, h, t)}(r, s, t) g(\theta^-(r, s, h, t), -1) |h, l(r, s, h, t), t\rangle, \quad (2.44)$$

$$|S(r, s, h, t)\rangle^+ = q(1 - rs, r - \theta^+(r, s, h, t) - 1) \mathcal{E}^{+, \theta^+(r, s, h, t)}(r, s, t) q(-\theta^+(r, s, h, t), 0) |h, l(r, s, h, t), t\rangle, \quad (2.45)$$

where  $\mathcal{E}^{\pm, \theta}(r, s, t)$  are the topological singular vector operators subjected to the spectral flow transform with parameter  $\theta$ , and

$$\begin{aligned} \theta^-(r, s, h, t) &= \frac{t}{2}(h - h^-(r, s, t)), \\ \theta^+(r, s, h, t) &= \frac{t}{2}(h - 1 - h^+(r, s, t)). \end{aligned} \quad (2.46)$$

The RHSs of (2.44) and (2.45) evaluate as elements of  $\mathcal{U}_{h,l(r,s,h,t),t}$  and satisfy the twisted massive highest-weight conditions

$$\begin{aligned} \mathcal{Q}_{\geq 1 \mp rs} |S(r, s, h, t)\rangle^\pm &= \mathcal{H}_{\geq 1} |S(r, s, h, t)\rangle^\pm = \mathcal{L}_{\geq 1} |S(r, s, h, t)\rangle^\pm = \mathcal{G}_{\geq \pm rs} |S(r, s, h, t)\rangle^\pm = 0, \\ \mathcal{L}_0 |S(r, s, h, t)\rangle^\pm &= l^\pm(r, s, h, t) |S(r, s, h, t)\rangle^\pm, \\ \mathcal{H}_0 |S(r, s, h, t)\rangle^\pm &= (h \mp rs) |S(r, s, h, t)\rangle^\pm \end{aligned} \quad (2.47)$$

with

$$l^\pm(r, s, h, t) = l(r, s, h, t) + \frac{1}{2}rs(rs + 2 \mp 1). \quad (2.48)$$

Either of the  $|S(r, s, h, t)\rangle^\pm$  states generates the entire massive Verma submodule; in particular, all of the dense  $\mathcal{G}/\mathcal{Q}$ -descendants of (2.44) and (2.45) are on the same extremal subdiagram (the extremal diagram of the submodule) and coincide up to numerical factors whenever they are in the same grade:

$$\begin{aligned} c_-(i, h, t) \mathcal{Q}_{i+1-rs} \dots \mathcal{Q}_{rs} |S(r, s, h, t)\rangle^- &= c_+(i, h, t) \mathcal{G}_{rs-i} \dots \mathcal{G}_{rs-1} |S(r, s, h, t)\rangle^+, \quad i = 0, \dots, 2rs, \\ c_-(i, h, t) \mathcal{G}_{-rs+i} \dots \mathcal{G}_{-rs-1} |S(r, s, h, t)\rangle^- &= c_+(i, h, t) \mathcal{G}_{-rs+i} \dots \mathcal{G}_{rs-1} |S(r, s, h, t)\rangle^+, \quad i \leq -1, \\ c_-(i, h, t) \mathcal{Q}_{rs-i} \dots \mathcal{Q}_{rs} |S(r, s, h, t)\rangle^- &= c_+(i, h, t) \mathcal{Q}_{rs-i} \dots \mathcal{Q}_{-rs} |S(r, s, h, t)\rangle^+, \quad i \geq 2rs + 1, \end{aligned} \quad (2.49)$$

where  $c_\pm(i, h, t)$  are ( $r$ - and  $s$ -dependent) polynomials in  $h$  and  $t$ .

PROOF. A state  $|h', t; \theta'\rangle_{\text{top}}$  or  $|h'', t; \theta''\rangle_{\text{top}}$  from (2.38) admits a singular vector if and only if (2.32) holds for the corresponding  $h'$  or  $h''$  parameter determined according to (2.29) and (2.31). Using (2.37), we see that this is the case if and only if  $\ell = l(r, s, h, t)$ ,  $r, s \in \mathbb{N}$ , which gives zeros of the Kač determinant [BFK]. Excluding the set  $\mathbb{X}(r, s, t)$  guarantees that this is the only zero. Further, a unique submodule can also occur for negative rational  $t = -\frac{\tilde{p}}{q}$  provided  $r$  is sufficiently small (the ‘smallness’ of  $r$  depends on whether  $s = 1$  or  $s \geq 1$ , since these cases correspond to different degenerations of the auxiliary topological Verma modules; the corresponding embedding diagrams are isomorphic [FST] to embedding diagrams of the  $\widehat{sl}(2)$  Verma modules with negative rational  $k + 2 = -\frac{\tilde{p}}{q}$  and with the same  $r$  and  $s$ ), whence Part I follows.

The fact that (2.44) and (2.45) are elements of the Verma module  $\mathcal{U}_{h,l(r,s,h,t),t}$  follows similarly to how this was described in the proof of Theorem 2.11 (in the present case, one consider the topological singular vector operators  $\mathcal{E}^\pm(r, s, t)$  as already expressed as polynomials in the modes, then one subjects these operators to the spectral flow transform with  $\theta = \theta^\pm(r, s, h, t)$ , and, finally, applies the formulae of Sec 2.2). Formulae (2.47) follow from (2.26)–(2.30) and (2.46). Equations (2.48) are obtained by applying (2.19) to explicit expressions (2.44) and (2.45). Two singular vectors (2.44) and (2.45) generate the same submodule because they are descendants of each other, as expressed by Eqs. (2.49), which, in turn, follows by comparing with the theory of  $\widehat{sl}(2)$  relaxed Verma modules by means of the direct and the inverse functors constructed in [FST].  $\square$

The structure of (2.44) and (2.45) reflects the property stipulated in item 2 of the list on page 15, that the corresponding topological highest-weight state from (2.38) admit a singular vector. Namely, Eqs. (2.44) and (2.45) mean that we first map from the massive Verma module  $\mathcal{U}_{h,l(r,s,h,t),t}$  either by  $g(\theta^-(r, s, h, t), -1)$  or by  $q(-\theta^+(r, s, h, t), 0)$  in such a way that the resulting state satisfies twisted *topological* highest-weight conditions with the twist parameters  $\theta^\mp(r, s, h, t)$  respectively, which are the roots of (2.37) with  $\ell = l(r, s, h, t)$ . Even though  $\theta^\mp(r, s, h, t)$  are, in general, complex, we build spectral-flow-transformed topological singular vectors on these states and, finally, map back to the original module  $\mathcal{U}_{h,l(r,s,h,t),t}$ .

From the correspondence with the zeros of the Kač determinant, we also see that the massive Verma module  $\mathcal{U}_{h,\ell,t}$  is irreducible if and only if conditions of item 1 of the list on p. 15 are satisfied.

### 3 Submodules and singular vectors in codimension $\geq 2$

To proceed with the degeneration patterns of  $N=2$  Verma modules, we begin with topological Verma modules, where 2 is the highest codimension, and then consider codimensions 2 and 3 in the massive case.

#### 3.1 Topological Verma modules

A further degeneration in the setting of Theorem 2.11 means that the parameter  $t$  is rational,  $t = p/q$ . This case is the least interesting one *as regards the structure of submodules*, since the structure of topological Verma module  $\mathcal{V}_{h^\pm(r,s,\frac{p}{q}),\frac{p}{q}}$  is determined [FST] by the well-known structure of the Verma module  $\mathcal{M}_{j^\pm(r,s,\frac{p}{q}-2),\frac{p}{q}-2}$  over the affine  $\widehat{sl}(2)$  algebra, where  $j^+(r,s,k) = \frac{r-1}{2} - (k+2)\frac{s-1}{2}$  and  $j^-(r,s,k) = -\frac{r+1}{2} + (k+2)\frac{s}{2}$ . This applies to the BGG resolution [BGG], embedding diagrams [RCW, Ma], etc.

Recall that the Verma module  $\mathcal{M}_{j,k}$  over the  $\widehat{sl}(2)$  algebra

$$\begin{aligned} [J_m^0, J_n^\pm] &= \pm J_{m+n}^\pm, & [J_m^0, J_n^0] &= \frac{K}{2} m \delta_{m+n,0}, \\ [J_m^+, J_n^-] &= K m \delta_{m+n,0} + 2J_{m+n}^0, & m, n &\in \mathbb{Z} \end{aligned} \quad (3.1)$$

(where the generator  $K$  is central) is freely generated by the modes  $J_{\leq -1}^+$ ,  $J_{\leq 0}^-$ , and  $J_{\leq -1}^0$  from the highest-weight vector  $|j, k\rangle_{sl(2)}$  that satisfies the following highest-weight conditions:

$$\begin{aligned} J_{\geq 0}^+ |j, k\rangle_{sl(2)} &= J_{\geq 1}^0 |j, k\rangle_{sl(2)} = J_{\geq 1}^- |j, k\rangle_{sl(2)} = 0, \\ J_0^0 |j, k\rangle_{sl(2)} &= j |j, k\rangle_{sl(2)}, & K |j, k\rangle_{sl(2)} &= k |j, k\rangle_{sl(2)}, \end{aligned} \quad j, k \in \mathbb{C}. \quad (3.2)$$

Singular vectors in  $\mathcal{M}_{j,k}$  are labelled by  $r, s \in \mathbb{N}$  and can be of the ‘+’ or ‘-’ type. General formulae for these singular vectors, which we denote as  $|MFF(r, s, k)\rangle^\pm$ , can be found in [MFF] or, in our present conventions, in [FST].

**Theorem 3.1 ([FST])** *For arbitrary  $h \in \mathbb{C}$  and  $t \in \mathbb{C} \setminus \{0\}$ ,*

1. *the topological  $N=2$  Verma module  $\mathcal{V}_{h,t}$  is irreducible if and only if the  $\widehat{sl}(2)$  Verma module  $\mathcal{M}_{-\frac{t}{2}h, t-2}$  is irreducible;*
2. *the module  $\mathcal{V}_{h,t}$  has a submodule generated by a singular vector  $|E(r, s, t)\rangle^\pm$ , Eqs. (2.33) or (2.34), if and only if the module  $\mathcal{M}_{-\frac{t}{2}h, t-2}$  has a submodule generated by the singular vector  $|MFF(r, s, t-2)\rangle^\pm$  respectively.*

Whenever the singular vector in  $\mathcal{M}_{-\frac{t}{2}h, t-2}$  has relative  $J_0^0$ -charge  $\pm r$ , it is clear from formulae (2.33) and (2.34) that the corresponding topological singular vector in  $\mathcal{V}_{h,t}$  has relative charge  $\mp r$  and satisfies the twisted topological highest-weight conditions (2.4) with the twist parameter  $\theta = \pm r$ .

Thus, the appearance of one or more singular vectors in a topological  $N=2$  Verma module can be read off from the corresponding  $\widehat{sl}(2)$  Verma module. In view of the correspondence at the level of *submodules*, it might seem puzzling that one can talk about subsingular vectors in topological  $N=2$  Verma modules, since these are absent in  $\widehat{sl}(2)$  Verma modules. In fact, this apparent paradox illustrates our general statement that, for the  $N=2$  superconformal algebra, subsingular vectors are an artifact of an “inconvenient” definition of singular vectors. They have to be considered when one restricts oneself to submodules generated only from the conventional, top-level,  $N=2$  singular vectors, which do not always generate maximal submodules. On the other hand, singular vectors (2.33) and (2.34), which satisfy *twisted*

topological highest-weight conditions, allow one to work with maximal submodules, and it is these singular vectors that are in 1 : 1 correspondence [S1, FST] with the  $\widehat{sl}(2)$  singular vectors.

In the conventional approach, on the other hand, subsingular vectors occur in the topological Verma modules  $\mathcal{V}_{h^\pm(r,s,t(r,s,n)),t(r,s,n)}$ , where  $t(r,s,n) = \frac{n-r}{s}$  with  $r$  greater than  $n$ . Namely, we have the following

**Proposition 3.2** *The quotient of the topological Verma module  $\mathcal{V}_{h,t}$  over the submodules generated by conventional singular vectors is reducible — i.e., a subsingular vector exists in  $\mathcal{V}_{h,t}$  — if and only if  $t = t(r,s,n)$ ,  $h = h^\pm(r,s,t(r,s,n))$ ,  $1 \leq n < r$ . In the ‘-’ case, for definiteness, the subsingular vector in  $\mathcal{V}_{h^-(r,s,t(r,s,n)),t(r,s,n)}$  is given by*

$$|\text{Sub}\rangle = \mathcal{G}_0 \dots \mathcal{G}_{r-n-1} \mathcal{G}_{r-n+1} \dots \mathcal{G}_{r-1} |E(r,s,\frac{n-r}{s})\rangle^- \quad (3.3)$$

(where  $|E(r,s,t)\rangle^-$  is the topological singular vector (2.34)). This becomes singular in the quotient module  $\mathcal{V}_{h^-(r,s,t(r,s,n)),t(r,s,n)}/\mathcal{C}$ , where  $\mathcal{C}$  is the submodule generated from the top-level representative of the singular vector in  $\mathcal{V}_{h^-(r,s,t(r,s,n)),t(r,s,n)}$ , which is given by  $\mathcal{G}_0 \dots \mathcal{G}_{r-1} |E(r,s,\frac{n-r}{s})\rangle^-$ .

PROOF. The parameters are such that the module  $\mathcal{V}_{h^-(r,s,t(r,s,n)),t(r,s,n)}$  contains at least two submodules

$$\mathcal{C}' \xrightarrow{|E(n,1,\frac{n-r}{s})\rangle^{+,r}} \mathcal{C} \xrightarrow{|E(r,s,\frac{n-r}{s})\rangle^-} \mathcal{V}_{h^-(r,s,t(r,s,n)),t(r,s,n)}, \quad (3.4)$$

where  $\mathcal{C} \approx \mathcal{V}_{h^-(r,s,t(r,s,n))-\frac{2r}{t},t(r,s,n);r}$  and  $\mathcal{C}' \approx \mathcal{V}_{h^-(r,s,t(r,s,n))+\frac{2(n-r)}{t},t(r,s,n);r-n}$ , the arrows mean embeddings by means of the corresponding singular vectors, and  $|E(r,s,t)\rangle^{\pm,\theta}$  are the topological singular vectors subjected to the spectral flow transform with parameter  $\theta$ . All we have to do in the conventional approach is to describe these submodules in terms of (submodules generated from) the top-level representatives of extremal diagrams. Thus, consider the conventional singular vector

$$|\text{conv}\rangle = \mathcal{G}_0 \dots \mathcal{G}_{r-1} |E(r,s,\frac{n-r}{s})\rangle^- \quad (3.5)$$

which satisfies highest-weight conditions (2.8) with  $\theta = 0$ . It is clear that  $|\text{conv}\rangle$  belongs to the submodule  $\mathcal{C}'$  iff  $n < r$  and belongs to  $\mathcal{C}$  iff  $n > r$ . Indeed, the highest-weight vector of  $\mathcal{C}'$  is  $|\text{h.w.}'\rangle = \mathcal{G}_{r-n} \dots \mathcal{G}_{r-1} |E(r,s,\frac{n-r}{s})\rangle^-$  and we have  $|\text{conv}\rangle = \mathcal{G}_0 \dots \mathcal{G}_{r-n-1} |\text{h.w.}'\rangle$  whenever  $n < r$ ; on the other hand,  $|\text{h.w.}'\rangle = \mathcal{G}_{r-n} \dots \mathcal{G}_{r-1} |\text{conv}\rangle$  whenever  $n > r$ . Hence, in the case where  $n < r$ , the module  $\mathcal{C}'$  is generated from  $|\text{conv}\rangle$  by the action of the  $N=2$  generators, whereas in the  $n > r$  case,  $|\text{conv}\rangle$  generates  $\mathcal{C}$ . There exists the minimal submodule  $\mathcal{N}$  such that  $\mathcal{C} \subset \mathcal{N}$  (it is possible that  $\mathcal{N} \approx \mathcal{V}_{h^-(r,s,t(r,s,n)),t(r,s,n)}$ ). Now, let us take the quotient of  $\mathcal{N}$  over the submodule generated from all conventional singular vectors. In the case where  $n > r$ , this is the quotient over the maximal submodule of  $\mathcal{N}$ , therefore the quotient is irreducible and, thus, there are no subsingular vectors in  $\mathcal{V}_{h^-(r,s,t(r,s,n)),t(r,s,n)}$ . On the other hand, in the case where  $n < r$ , this quotient contains the highest-weight vector of  $\mathcal{C}$  and, therefore, is reducible.

The explicit formula (3.3) is obvious from the analysis of extremal diagrams (see (3.7) and below), however it can also be checked by direct calculations that the vector (3.3) satisfies conventional highest-weight conditions (Eqs. (2.8) with  $\theta = 0$ ) modulo descendants of  $|\text{conv}\rangle$ :

$$\mathcal{H}_1 |\text{Sub}\rangle = \mathcal{G}_0 \dots \mathcal{G}_{r-n-2} \mathcal{G}_{r-n} \dots \mathcal{G}_{r-1} |E(r,s,\frac{n-r}{s})\rangle^- = (-1)^{(r-n-1)} \frac{r-n}{2s(n+1)} \mathcal{Q}_{-r+n+1} |\text{conv}\rangle, \quad (3.6)$$

and similarly for the other annihilation conditions. Finally, for modules  $\mathcal{V}_{h,t}$  with  $h$  and  $t$  not as in the Proposition, the topological singular vectors and the conventional one are dense  $\mathcal{G}/\mathcal{Q}$ -descendants of each other. Therefore, they generate the same submodule and there are no subsingular vectors in those cases.  $\square$



### 3.2 Massive singular vectors in codimension 2

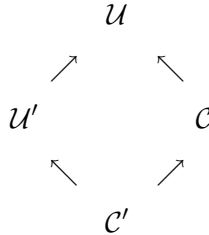
We now turn to massive Verma modules. In codimension 2, three cases from the list on page 15 are arranged into the three following Theorems (3.4, 3.6, and 3.8), while Propositions 3.5, 3.7, and 3.9 are given in order to make contact with the conventional description in terms of top-level, untwisted, representatives of singular vectors and, accordingly, in terms of subsingular vectors; we show why the subsingular vectors appear and how they can be constructed explicitly.

The following observations are central for the subsequent constructions:

**Lemma 3.3** *Let  $\mathcal{U} \equiv \mathcal{U}_{h,\ell,t}$  be a massive Verma module.*

- i) *Let  $\mathcal{U} \supset \mathcal{U}'$  and  $\mathcal{U} \supset \mathcal{C}$ , where  $\mathcal{U}'$  is a massive Verma submodule and  $\mathcal{C}$  is a twisted topological Verma module generated from a charged singular vector in  $\mathcal{U}$  such that for any twisted topological Verma module  $\mathcal{C}''$ ,  $\mathcal{U} \supset \mathcal{C}'' \supset \mathcal{C}$ , it follows that  $\mathcal{C}'' = \mathcal{C}$ . Then there exists a twisted topological Verma module  $\mathcal{C}' = \mathcal{U}' \cap \mathcal{C} \neq \{0\}$ . Moreover, the embeddings  $\mathcal{C}' \subset \mathcal{C}$  and  $\mathcal{C}' \subset \mathcal{U}'$  are given by a topological singular vector in  $\mathcal{C}$  and by a charged singular vector in  $\mathcal{U}'$  respectively.*
- ii) *Conversely, if  $\mathcal{U} \supset \mathcal{U}'$ , where  $\mathcal{U}'$  is a massive Verma module, and  $\mathcal{U}' \supset \mathcal{C}'$ , where  $\mathcal{C}'$  is a submodule generated from a charged singular vector in  $\mathcal{U}'$ , then  $\mathcal{U} \supset \mathcal{C}$ , where  $\mathcal{C}$  is a submodule generated from a charged singular vector in  $\mathcal{U}$ . Moreover,  $\mathcal{C}$  is maximal ( $\mathcal{U} \supset \mathcal{C}'' \supset \mathcal{C} \implies \mathcal{C}'' = \mathcal{C}$ ),  $\mathcal{C}' \subset \mathcal{U}' \cap \mathcal{C}$ , and  $\mathcal{C}'$  is generated from a topological singular vector in the topological Verma module  $\mathcal{C}$ .*
- iii) *If  $\mathcal{U} \supset \mathcal{C}'$ , where  $\mathcal{C}'$  is a twisted topological Verma module, there exists a twisted topological Verma submodule  $\mathcal{C} \subset \mathcal{U}$  such that the embedding is given by the charged singular vector,  $\mathcal{C}$  is maximal ( $\mathcal{U} \supset \mathcal{C}'' \supset \mathcal{C} \implies \mathcal{C}'' = \mathcal{C}$ ), and  $\mathcal{C}' \subset \mathcal{C}$  (with the embedding given by a topological singular vector).*

PROOF. The Lemma can be illustrated by



As regards item i), let us assume the contrary, namely that  $\mathcal{U}' \cap \mathcal{C} = \{0\}$ . We then take the quotient  $\mathcal{Q} = \mathcal{U}/\mathcal{C}$ , which is a twisted topological Verma module. It should contain all of the extremal states of the massive Verma module  $\mathcal{U}'$ , however some of these states are clearly outside the extremal diagram of  $\mathcal{Q}$  according to their bigrading. Thus,  $\mathcal{U}' \cap \mathcal{C} = \mathcal{C}' \neq \{0\}$ .

Further, it follows from Theorem 2.4 that the embedding  $\mathcal{C}' \subset \mathcal{C}$  is given by a topological singular vector in  $\mathcal{C}$ . In the extremal diagram of  $\mathcal{U}'$ , choose a state  $|\star'\rangle$  from which all of the  $\mathcal{U}'$  module is generated and consider the dense  $\mathcal{G}/\mathcal{Q}$ -descendant  $|\text{top}'\rangle$  of  $|\star'\rangle$  with the minimal number of the  $\mathcal{G}$  or  $\mathcal{Q}$  modes among those dense  $\mathcal{G}/\mathcal{Q}$ -descendants that belong to  $\mathcal{C}'$ . Such a state necessarily exists, since otherwise the quotient  $\mathcal{U}/\mathcal{C}$  would contain extremal states of  $\mathcal{U}'$  that lie outside the module  $\mathcal{U}/\mathcal{C}$ . The state  $|\text{top}'\rangle$  satisfies the twisted topological highest-weight conditions and the module  $\mathcal{C}'$  is generated from  $|\text{top}'\rangle$ . Therefore,  $|\text{top}'\rangle$  coincides with a topological singular vector in  $\mathcal{C}$  and is at the same time a charged singular vector in  $\mathcal{U}'$ . This completes the proof of i).

To prove ii), let us fix extremal states  $|\star\rangle$  in  $\mathcal{U}$  and  $|\star'\rangle$  in  $\mathcal{U}'$  such that  $\mathcal{U}$  and  $\mathcal{U}'$  are generated from  $|\star\rangle$  and  $|\star'\rangle$  respectively. The highest-weight vector  $|\text{top}'\rangle$  of  $\mathcal{C}'$  is a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of  $|\star'\rangle$ . Now, a charged

singular vector exists in the extremal diagram of the submodule whenever  $l^\pm(r, s, h, t) = l_{\text{ch}}(N, h \mp rs, t)$  (see Eqs. (2.39) and (2.48)), which implies a similar relation for the dimension  $l(r, s, h, t)$  of the massive Verma module  $\mathcal{U}_{h, l(r, s, h, t), t}$ . Assuming, for definiteness, that  $|\text{top}'\rangle$  is a dense  $\mathcal{G}$ -descendant of  $|\star'\rangle$ , we thus see that there exists a state  $|\text{top}\rangle$  such that it is a dense  $\mathcal{G}$ -descendant of  $|\star\rangle$ , satisfies twisted topological highest-weight conditions, and is not a dense  $\mathcal{G}$ -descendant of any other state satisfying twisted topological highest-weight conditions. Let us consider the module  $\mathcal{C}$  generated from  $|\text{top}\rangle$  and take the quotient of  $\mathcal{Q} = \mathcal{U}/\mathcal{C}$ , which is a twisted topological Verma module. By analyzing the bigradings, it is easy to see that some of the dense  $\mathcal{G}$ -descendants of  $|\star'\rangle$  lie outside the extremal diagram of  $\mathcal{Q}$ , therefore these dense  $\mathcal{G}$ -descendants belong to  $\mathcal{C}$ . Therefore,  $\mathcal{C} \cap \mathcal{C}' \neq \{0\}$ . If  $\mathcal{C} \cap \mathcal{C}' \neq \mathcal{C}'$ , the module  $\mathcal{Q}$  contains the submodule  $\mathcal{C}'/(\mathcal{C} \cap \mathcal{C}')$  which is not a twisted topological Verma module. This contradicts Theorem 2.4. Therefore,  $\mathcal{C} \cap \mathcal{C}' = \mathcal{C}'$  and  $\mathcal{C}' \subset \mathcal{U}' \cap \mathcal{C}$ , whence also follows the fact that the embedding  $\mathcal{C}' \subset \mathcal{C}$  is given by the topological singular vector.

The proof of iii), which is rather tedious, is relegated to the Appendix.  $\square$

The Lemma is used in the following Theorem, which describes the occurrence of (at least!) two different massive singular vectors in the massive Verma module  $\mathcal{U}_{h, \ell, t}$ . This is case 4 of the list on page 15 and it corresponds to rational  $t$ :

**Theorem 3.4** *The highest-weight of the massive Verma module  $\mathcal{U}_{h, \ell, t}$  belongs to the set  $\mathcal{O}_{\text{mm}}$  if and only if  $\ell = l(r, s, h, t)$ , where*

$$(r, s, h, t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{C} \times \mathbb{Q} \setminus \left( \left\{ (r, s, \pm s - \frac{2n-1 \pm r}{p/q}, \frac{p}{q}) \mid r, s \in \mathbb{Z}, n \in \mathbb{Z}, p \in \mathbb{Z}, q \in \mathbb{N} \right\} \cup \mathbb{Y} \right)$$

with  $\mathbb{Y}$  as in (2.43). Then,

1. any primitive submodule of  $\mathcal{U}_{h, \ell, t}$  is a massive Verma module generated from the representative  $|S(a, b, h, \frac{p}{q})\rangle^-$  of a massive singular vector, where  $a, b \in \mathbb{N}$  is a solution to  $\ell = l(a, b, h, \frac{p}{q})$ ; equivalently, that submodule is also generated from the  $|S(a, b, h, \frac{p}{q})\rangle^+$  representative of the massive singular vector with the same  $a$  and  $b$ .
2. The structure of  $\mathcal{U}_{h, l(r, s, h, \frac{p}{q}), \frac{p}{q}}$  is determined by the structure of the topological Verma module  $\mathcal{V}_{h^-(r, s, \frac{p}{q}), \frac{p}{q}}$  in the following way:
  - (a) for any massive Verma submodule  $\mathcal{U}' \subset \mathcal{U}_{h, l(r, s, h, \frac{p}{q}), \frac{p}{q}}$  generated from a massive singular vector, there exists a submodule in  $\mathcal{V}_{h^-(r, s, \frac{p}{q}), \frac{p}{q}}$  generated from a topological singular vector  $e = \mathcal{E}^\pm(a, b, \frac{p}{q}) |h^-(r, s, \frac{p}{q}), \frac{p}{q}\rangle_{\text{top}}$ ,  $a, b \in \mathbb{N}$ , such that  $\mathcal{U}'$  is generated from the massive singular vector
$$g(-ab, \mp a + \theta^-(a, b, h, \frac{p}{q}) - 1) \mathcal{E}^{\pm, \theta^-(a, b, h, \frac{p}{q})}(a, b, \frac{p}{q}) g(\theta^-(a, b, h, \frac{p}{q}), -1) \left| h, l(r, s, h, \frac{p}{q}), \frac{p}{q} \right\rangle; \quad (3.8)$$
  - (b) conversely, for any singular vector in  $\mathcal{V}_{h^-(r, s, \frac{p}{q}), \frac{p}{q}}$  of the form  $e = |E(a, b, \frac{p}{q})\rangle^+$  with  $a \geq 1$ ,  $b \geq 2$ , or  $e = |E(a, b, \frac{p}{q})\rangle^-$  with  $a, b \geq 1$ , there exists a massive singular vector constructed as in (3.8) that generates a massive Verma submodule  $\mathcal{U}' \subset \mathcal{U}_{h, l(r, s, h, \frac{p}{q}), \frac{p}{q}}$ ;
  - (c) two different singular vectors  $e_1$  and  $e_2$  in  $\mathcal{V}_{h^-(r, s, \frac{p}{q}), \frac{p}{q}}$  correspond in this way to the same massive Verma submodule  $\mathcal{U}' \subset \mathcal{U}_{h, l(r, s, h, \frac{p}{q}), \frac{p}{q}}$  if and only if one of the  $e_i$  is the  $|E(c \geq 1, 1, \frac{p}{q})\rangle^+$  singular vector in the module generated from the other.

(As before,  $\mathcal{E}^{\pm, \theta}(r, s, t)$  is the spectral flow transform of the topological singular vector operator read off from (2.33) and (2.34); *primitive* refers to a submodule that is not a sum of other submodules.)

PROOF. By definition, the highest-weight of the Verma module  $\mathcal{U}_{h,\ell,t}$  belongs to  $\mathcal{O}_{\text{mm}}$  whenever each of the states (2.38) admits two topological singular vectors and none of the solutions to Eq. (2.37) is an integer:  $\theta', \theta'' \notin \mathbb{Z}$ . The last condition reformulates as the constraint  $\frac{1}{2} \left( 1 - ht \pm \sqrt{4\ell t + (ht - 1)^2} \right) \notin \mathbb{Z}$ . On the other hand, analysing all possible cases where the states (2.38) have the specified number of singular vectors for negative rational  $t = -\frac{p}{q}$  requires analysing the embedding diagrams of the auxiliary topological Verma modules, these embedding diagrams being isomorphic to the standard embedding diagrams of  $\widehat{\mathfrak{sl}}(2)$  Verma modules. This gives the values of  $(r, s, h, t)$  as in the Theorem. Further, by the definition of  $\mathcal{O}_{\text{mm}}$ , there are no charged singular vectors in  $\mathcal{U}_{h,\ell,t}$ , therefore taking into account Lemma 3.3 we obtain that each submodule is generated from (2.44) as well as from (2.45).

Part 2 becomes obvious from the explicit formulae for massive singular vectors (2.44) and (2.45). Let us consider, for definiteness, Eq. (2.44). The part  $g(\theta^-(r, s, h, \frac{p}{q}), -1) \cdot |h, l(r, s, h, \frac{p}{q}), \frac{p}{q}\rangle$  of the formula represents the highest-weight vector of the twisted topological Verma module  $\mathfrak{V}_{h^-(r, s, \frac{p}{q}), \frac{p}{q}; \theta^-(r, s, h, \frac{p}{q})}$ . Now, let us take any element  $|\nu\rangle$  from the extremal diagram of the massive submodule which satisfies the twisted massive highest-weight conditions with the twist  $\theta = \nu$ . The operator  $g(\mp a + \theta^-(r, s, h, \frac{p}{q}), \nu - 1)$  maps the state  $|\nu\rangle$  into the module  $\mathfrak{V}_{h^-(r, s, \frac{p}{q}), \frac{p}{q}; \theta^-(r, s, h, \frac{p}{q})}$ . The image of  $|\nu\rangle$  under this mapping is a twisted topological singular vector referred to in Part 2(a). The fact that  $e_1$  is the  $|E(c \geq 1, 1, \frac{p}{q})^+$  singular vector built on  $e_2 = |E(a, b, \frac{p}{q})^\pm$  means that  $e_1 = g(\mp a - c, \mp a - 1) |E(a, b, \frac{p}{q})^\pm$ . Then Part 2(c) follows from the identity

$$\begin{aligned} & g(-ab, a - c + \theta^-(a, b, h, \frac{p}{q}) - 1) \cdot \\ & \cdot g(\mp a - c + \theta^-(a, b, h, \frac{p}{q}), \mp a + \theta^-(a, b, h, \frac{p}{q}) - 1) \mathcal{E}^{\pm, \theta^-(a, b, h, \frac{p}{q})}(a, b, \frac{p}{q}) \cdot \\ & g(\theta^-(a, b, h, \frac{p}{q}), -1) \left| h, l(r, s, h, \frac{p}{q}), \frac{p}{q} \right\rangle = \\ & g(-ab, a + \theta^-(a, b, h, \frac{p}{q}) - 1) \mathcal{E}^{\pm, \theta^-(a, b, h, \frac{p}{q})}(a, b, \frac{p}{q}) g(\theta^-(a, b, h, \frac{p}{q}), -1) \left| h, l(r, s, h, \frac{p}{q}), \frac{p}{q} \right\rangle, \end{aligned} \quad (3.9)$$

where we used Eqs. (2.13). Let us point out once again that the constructions of the type of  $g(-ab, \mp a + \theta - 1) \mathcal{E}^{\pm, \theta}(a, b, t) g(\theta, -1) |h, \ell, t\rangle$  in Eq. (3.8) evaluate as Verma module elements using the formulae of Sec. 2.2.  $\square$

If we recall that the structure of topological Verma modules is equivalent [FST] to the structure of the standard  $\widehat{\mathfrak{sl}}(2)$  Verma modules, we see that the modules  $\mathcal{U}_{h,\ell,t}$  with  $(h, \ell, t) \in \mathcal{O}_{\text{mm}}$ , too, have essentially the same (familiar) structure as the  $\widehat{\mathfrak{sl}}(2)$  Verma modules with a rational level  $k = t - 2$ .

In the present case, restricting oneself to only top-level singular vectors is innocuous<sup>8</sup>:

**Proposition 3.5** *Under the conditions of Theorem 3.4, the quotient of  $\mathcal{U}_{h, l(r, s, h, \frac{p}{q}), \frac{p}{q}}$  with respect to the conventional singular vectors is irreducible, i.e., no subsingular vectors exist in the massive Verma module  $\mathcal{U}_{h, l(r, s, h, \frac{p}{q}), \frac{p}{q}}$ .*

PROOF. Indeed, in this case there are no charged singular vectors in  $\mathcal{U}_{h, l(r, s, h, \frac{p}{q}), \frac{p}{q}}$ , therefore each of the extremal states of the submodule is a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of any other extremal state of the same submodule. Thus, each element of the extremal diagram of the submodule generates the same module.  $\square$

Next is the case where  $\mathcal{U}_{h,\ell,t}$  contains two charged singular vectors none of which are descendants of the other, i.e. the extremal diagram contains two states that satisfy twisted topological highest-weight

<sup>8</sup>We remind the reader that, when we are talking about subsingular vectors, these are understood in the setting where the conventional definition of singular vectors is adopted, i.e., only top-level representatives of extremal diagrams are ‘allowed’ to generate submodules.

conditions and lie on the different sides of the highest-weight vector. That is, the extremal diagram has two branching points similar to that in (2.10), but on the different sides of  $|h, \ell, t\rangle$ . This is case 5 in the list on page 15:

**Theorem 3.6** *The highest-weight of the massive Verma module  $\mathcal{U}_{h,\ell,t}$  belongs to the set  $\mathcal{O}_{cc}$  if and only if  $h = h_{cc}(n, m, t)$  and  $\ell = l_{cc}(n, m, t)$ , where*

$$h_{cc}(n, m, t) = \frac{1}{t}(1 - m - n), \quad l_{cc}(n, m, t) = -\frac{mn}{t}, \quad (3.10)$$

$$(n, m, t) \in \left( \mathbb{N} \times (-\mathbb{N}_0) \times (\mathbb{C} \setminus \mathbb{Q}) \right) \cup \left\{ (n', m', -\frac{p}{q}) \mid n' \in \mathbb{N}, m' \in -\mathbb{N}_0, p, q \in \mathbb{N}, 1 \leq n' - m' \leq q \right\}.$$

Then the massive Verma module  $\mathcal{U}_{h_{cc}(n,m,t), l_{cc}(n,m,t), t}$  contains two twisted topological Verma submodules  $\mathcal{C}_1 \approx \mathfrak{V}_{\frac{m-n-1}{t}, t; -m}$  and  $\mathcal{C}_2 \approx \mathfrak{V}_{\frac{n+1-m}{t}, t; -n}$  generated from the charged singular vectors  $|E(m, h_{cc}(n, m, t), t)\rangle_{ch}$  and  $|E(n, h_{cc}(n, m, t), t)\rangle_{ch}$  respectively. The maximal submodule in  $\mathcal{U}_{h_{cc}(n,m,t), l_{cc}(n,m,t), t}$  is  $\mathcal{C}_1 \cup \mathcal{C}_2$ , and  $\mathcal{C}_1 \cap \mathcal{C}_2 = 0$ .

PROOF. The definition of the set  $\mathcal{O}_{cc}$  implies that both solutions  $\theta'$  and  $\theta'' = -\theta' + ht - 1$  of equation (2.37) are integers, whence the conditions  $h = h_{cc}(n, m, t)$  and  $\ell = l_{cc}(n, m, t)$  follow. Then the singular vectors referred to in the theorem are all singular vectors in the module  $\mathcal{U}_{h,\ell,t}$ , since, in accordance with Lemma 3.3, any other submodule in  $\mathcal{U}_{h,\ell,t}$  would have non-empty intersections with  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , which would then be generated from singular vectors in  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . But by the definition of the set  $\mathcal{O}_{cc}$ , the modules  $\mathcal{C}_1$  and  $\mathcal{C}_2$  both are irreducible.  $\square$

This case is still harmless if one wishes to work with only the top-level representatives of extremal diagrams of submodules. Since there are only two singular vectors in  $\mathcal{U}_{h_{cc}(n,m,t), l_{cc}(n,m,t), t}$ , we immediately obtain

**Proposition 3.7** *Under the conditions of Theorem 3.6, the quotient of  $\mathcal{U}_{h_{cc}(n,m,t), l_{cc}(n,m,t), t}$  with respect to the conventional singular vectors is irreducible, i.e., there are no subsingular vectors in the massive Verma module  $\mathcal{U}_{h_{cc}(n,m,t), l_{cc}(n,m,t), t}$ .*

The third possibility in codimension 2, as described in case 6 of the list on page 15, is when one of the states (2.38) belongs to the original module  $\mathcal{U}_{h,\ell,t}$ , hence there is a charged singular vector in  $\mathcal{U}_{h,\ell,t}$ . The submodule  $\mathcal{C}$  generated from the charged singular vector contains a singular vector. One of the possibilities is that this is a second charged singular vector in  $\mathcal{U}_{h,\ell,t}$ , situated on the same side from the highest-weight vector as the first charged singular vector. Otherwise, the submodule generated from the singular vector in  $\mathcal{C}$  corresponds to a massive Verma submodule in  $\mathcal{U}_{h,\ell,t}$  in accordance with Lemma 3.3.

**Theorem 3.8** *The highest-weight of the massive Verma module  $\mathcal{U}_{h,\ell,t}$  belongs to the set  $\mathcal{O}_{cm}$  if and only if  $h = h_{cm}^\sigma(r, s, n, t)$ ,  $\ell = l_{cm}^\sigma(r, s, n, t)$ , where  $\sigma \in \{-, +\}$ ,*

$$h_{cm}^\pm(r, s, n, t) = \frac{1-2n}{t} \pm (s - \frac{r}{t}), \quad l_{cm}^\pm(r, s, n, t) = \frac{n}{t}(-n \pm (st - r)), \quad (3.11)$$

$$(\sigma, r, s, n, t) \in \left( \{\pm\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z} \times (\mathbb{C} \setminus \mathbb{Q}) \right) \cup \mathbb{A} \cup \mathbb{B},$$

where

$$\mathbb{A} = \left\{ (\{+\}, r, 0, n, t) \mid r \in \mathbb{N}, n \in \mathbb{N}, t \in \mathbb{C} \setminus \mathbb{Q} \right\} \cup \left\{ (\{-\}, r, 0, n, t) \mid r \in \mathbb{N}, n \in -\mathbb{N}_0, t \in \mathbb{C} \setminus \mathbb{Q} \right\},$$

$$\mathbb{B} = \left\{ (\{+\}, r, s, n, -\frac{p}{q}) \mid n \in \mathbb{N}, p, q \in \mathbb{N}, 1 \leq r \leq p, 0 \leq s \leq q-1 \right\} \cup \left\{ (\{-\}, r, s, n, -\frac{p}{q}) \mid n \in -\mathbb{N}_0, p, q \in \mathbb{N}, 1 \leq r \leq p, 0 \leq s \leq q-1 \right\}. \quad (3.12)$$

Then,  $\mathcal{U}_{\mathbf{h}_{\text{cm}}^\pm(r,s,n,t), \mathbf{l}_{\text{cm}}^\pm(r,s,n,t), t}$  contains a twisted topological Verma submodule  $\mathcal{C} \approx \mathfrak{V}_{\mathbf{h}^\pm(r,s,t), t; -n}$  generated from the charged singular vector  $|E(n, \mathbf{h}_{\text{cm}}^\pm(r, s, n, t), t)\rangle_{\text{ch}}$ .

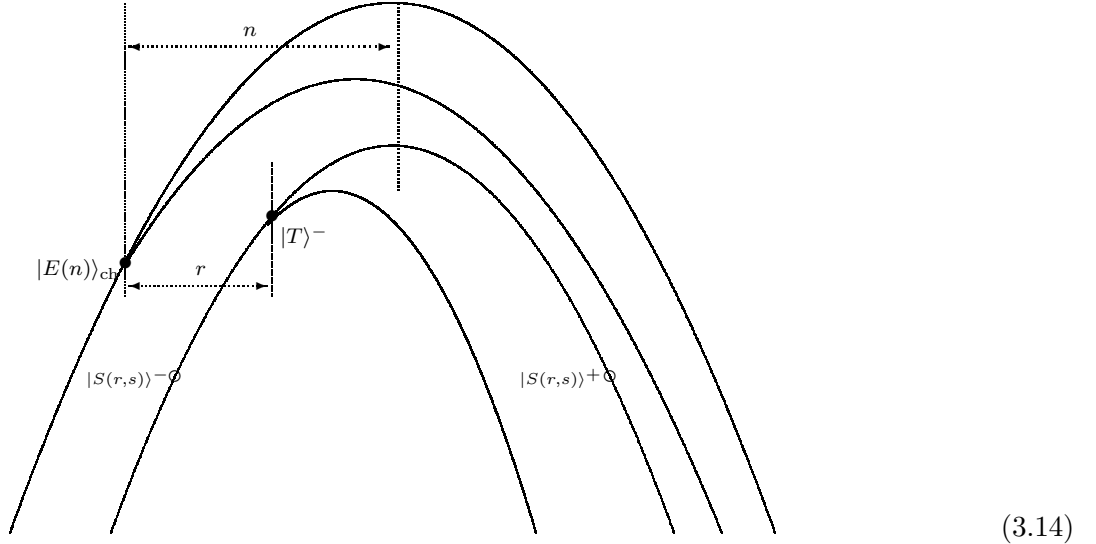
For  $s \neq 0$ , further,  $\mathcal{U}_{\mathbf{h}_{\text{cm}}^\pm(r,s,n,t), \mathbf{l}_{\text{cm}}^\pm(r,s,n,t), t}$  contains a massive Verma submodule  $\mathcal{U}'$  generated from the massive singular vector  $|S(r, s, \mathbf{h}_{\text{cm}}^\pm(r, s, n, t), t)\rangle^+$  (if  $n \geq 1$ ) or  $|S(r, s, \mathbf{h}_{\text{cm}}^\pm(r, s, n, t), t)\rangle^-$  (if  $n \leq 0$ ), where  $r, s \geq 1$ . The maximal submodule in  $\mathcal{U}_{\mathbf{h}_{\text{cm}}^\pm(r,s,n,t), \mathbf{l}_{\text{cm}}^\pm(r,s,n,t), t}$  is  $\mathcal{U}' \cup \mathcal{C}$ . The intersection  $\mathcal{U}' \cap \mathcal{C}$  is a twisted topological Verma module generated from the topological singular vector in  $\mathcal{C}$  given by

$$|T\rangle^\pm = \begin{cases} \mathcal{E}^{\pm, -n}(r, s + \frac{1}{2} \pm \frac{1}{2}, t) |E(n, \mathbf{h}_{\text{cm}}^\pm(r, s, n, t), t)\rangle_{\text{ch}}, & n \in \mathbb{N}, \\ \mathcal{E}^{\pm, -n}(r, s + \frac{1}{2} \mp \frac{1}{2}, t) |E(n, \mathbf{h}_{\text{cm}}^\pm(r, s, n, t), t)\rangle_{\text{ch}}, & n \in -\mathbb{N}_0. \end{cases} \quad (3.13)$$

When  $s = 0$ , the corresponding state (3.13) is a topological singular vector in  $\mathcal{C}$  and, at the same time, a charged singular vector in  $\mathcal{U}_{h, \ell, t}$ .

PROOF. The definition of  $\mathcal{O}_{\text{cm}}$  requires that precisely one of the solutions of Eq. (2.37) be an integer, whence the existence of submodule  $\mathcal{C}$  follows. Further, by the definition of  $\mathcal{O}_{\text{cm}}$ , the module  $\mathcal{C}$  contains precisely one singular vector. This is vector (3.13). If this vector were  $|E^+(r, 1, t)\rangle$  for  $n > 0$  or  $|E^-(r, 1, t)\rangle$  for  $n \leq 0$ , it would be a second charged singular vector in  $\mathcal{U}_{h, \ell, t}$ . Then, by the conditions of the Theorem and Lemma 3.3, there are no other submodules in  $\mathcal{U}_{h, \ell, t}$ . If singular vector (3.13) is not one of the above, we see from Lemma 3.3 that any other submodule in  $\mathcal{U}_{h, \ell, t}$  is a massive Verma module that has a nontrivial intersection with  $\mathcal{C}$ . However, any submodule in  $\mathcal{U}_{h, \ell, t}$  can intersect  $\mathcal{C}$  over the submodule generated from the only singular vector (3.13) in  $\mathcal{C}$ . Thus,  $\mathcal{U}_{h, \ell, t}$  can contain only one massive submodule. Finally, the state  $|S(r, s, \mathbf{h}_{\text{cm}}^\pm(r, s, n, t), t)\rangle^+$  in the case of  $n \geq 1$  or  $|S(r, s, \mathbf{h}_{\text{cm}}^\pm(r, s, n, t), t)\rangle^-$  in the case of  $n \leq 0$  generates this submodule because the respective state satisfies the (twisted) massive highest-weight conditions and does not belong to  $\mathcal{C}$  since  $\mathcal{C}$  contains no states with the gradings as that of the respective  $|S(\dots)\rangle^\pm$  state<sup>9</sup>.  $\square$

The situation described in the Theorem is illustrated in the following extremal diagram (choosing, for definiteness,  $n > 0$  and the ‘ $-$ ’ case in (3.11))



Here, the vector  $|S(r, s)\rangle^+ \equiv |S(r, s, \mathbf{h}_{\text{cm}}^-(r, s, n, t), t)\rangle^+$  is a representative of the massive singular vector in

<sup>9</sup>On the other hand, for  $n > 0$  for example, the vector  $|S(r, s, \mathbf{h}_{\text{cm}}^-(r, s, n, t), t)\rangle^-$  belongs to the topological Verma submodule  $\mathcal{C}'$  generated from the highest-weight state (3.13) whenever  $n \leq r(s+1)$ , in which case it is then a dense  $\mathcal{G}$ -descendant of  $|T\rangle^-$ :  $|S(r, s, \mathbf{h}_{\text{cm}}^-(r, s, n, t), t)\rangle^- = \mathcal{G}_{-rs} \dots \mathcal{G}_{r-n-1} |T\rangle^-$ , and similarly for  $n \leq 0$  with  $+\leftrightarrow -$ .

the sense of Definition 2.9, since its dense  $\mathcal{G}/\mathcal{Q}$ -descendants generate the entire extremal diagram of the massive Verma submodule  $\mathcal{U}'$ . According to (3.13), the twisted topological highest-weight state  $|T\rangle^-$  is the embedding of the topological singular vector  $|E(r, s, t)\rangle^-$  into the submodule built on the charged singular vector  $|E(n)\rangle_{\text{ch}} \equiv |E(n, h_{\text{cm}}^-(r, s, n, t), t)\rangle_{\text{ch}}$ . The diagram shows the case where  $n \leq r(s+1)$  and, thus, the vector  $|S(r, s)\rangle^- \equiv |S(r, s, h_{\text{cm}}^-(r, s, n, t), t)\rangle^-$  belongs to the topological Verma submodule  $\mathcal{C}'$  generated from the highest-weight state (3.13). In particular, its dense  $\mathcal{G}/\mathcal{Q}$ -descendants do not generate the same diagram as dense  $\mathcal{G}/\mathcal{Q}$ -descendants of  $|S(r, s, h_{\text{cm}}^-(r, s, n, t), t)\rangle^+$ . Together, the vectors  $|E(n, h_{\text{cm}}^-(r, s, n, t), t)\rangle_{\text{ch}}$  and  $|S(r, s, h_{\text{cm}}^-(r, s, n, t), t)\rangle^+$  generate a maximal submodule. The submodules generated by each of these vectors intersect over the submodule generated from  $|T\rangle^-$ .

In this case, when one wishes to work with only the conventional, top-level, representatives of singular vectors, one has to pay the price of considering subsingular vectors. Their positions and explicit constructions are a direct consequence of the above analysis. In the following proposition, we thus assume the conventional definition of singular vectors, demanding that these always satisfy the ‘untwisted’ highest-weight conditions. Then, the subsingular vectors are as follows:

**Proposition 3.9** *Under the conditions of Theorem 3.8, the massive Verma module  $\mathcal{U}_{h,\ell,t}$  contains a subsingular vector if and only if*

1. *either  $r \geq n > 0$  and  $h = h_{\text{cm}}^-(r, s, n, t)$ ,  $s \geq 1$ ,*
2. *or  $n \leq 0$ ,  $r \geq |n| + 1$ , and  $h = h_{\text{cm}}^+(r, s, n, t)$ ,  $s \geq 1$ .*

*In the first case, the subsingular vector is given by*

$$\begin{aligned} |\text{Sub}\rangle &= \mathcal{G}_0 \dots \mathcal{G}_{-n+r} q(1-r+n, n+t-1) \mathcal{E}^{+, -n+r-t}(r, s, t) q(n-r+t, 0) |h_{\text{cm}}^-(n, r, s, t), l_{\text{cm}}^-(n, r, s, t), t\rangle \\ &= \mathcal{G}_0 \dots \mathcal{G}_{r-n-1} \mathcal{G}_{r-n+1} \dots \mathcal{G}_{rs-1} |S(r, s, h_{\text{cm}}^-(n, r, s, t), t)\rangle^+. \end{aligned} \quad (3.15)$$

*This vector (which has the relative charge 1) becomes singular in the module obtained by taking the quotient over the submodule generated by the (top-level) singular vector*

$$|s\rangle = \mathcal{G}_0 \dots \mathcal{G}_{r-n-1} \mathcal{E}^{-, -n}(r, s, t) \mathcal{G}_{-n} \dots \mathcal{G}_{-1} |h_{\text{cm}}^-(n, r, s, t), l_{\text{cm}}^-(n, r, s, t), t\rangle. \quad (3.16)$$

*In the case where  $n \leq 0$ ,  $r \geq |n| + 1$ , similarly,*

$$|\text{Sub}\rangle = \mathcal{Q}_1 \dots \mathcal{Q}_{r+n-1} \mathcal{Q}_{r+n+1} \dots \mathcal{Q}_{rs} |S(r, s, h_{\text{cm}}^+(n, r, s, t), t)\rangle^-. \quad (3.17)$$

Let us remind the reader that, as in the general construction (2.44), (2.45) of  $N=2$  singular vectors, the state  $q(1-r+n, n+t-1) \mathcal{E}^{+, -n+r-t}(r, s, t) q(n-r+t, 0) |h, \ell, t\rangle$  in (3.15) evaluates as an element of  $\mathcal{U}_{h,\ell,t}$  using the formulae of Sec. 2.2, see also [ST2]. Recall also that, as before,  $\mathcal{E}^{\pm, \theta}(r, s, t)$  are topological singular vector operators transformed by the spectral flow with the parameter  $\theta$ .

PROOF. Consider, for definiteness, case 1 of the Proposition. Then, the module  $\mathcal{U}_{h,\ell,t}$  contains only three submodules  $\mathcal{C} \subset \mathcal{U}_{h,\ell,t}$ ,  $\mathcal{U}' \subset \mathcal{U}_{h,\ell,t}$ , and  $\mathcal{C}' \subset \mathcal{U}'$ ,  $\mathcal{C}' \subset \mathcal{C}$ , where  $\mathcal{C}$  and  $\mathcal{U}'$  are as in Theorem 3.8 and  $\mathcal{C}' \approx \mathfrak{V}_{h_{\text{cm}}^-(r, s, n, t) + \frac{2}{t}(n-r), t; r-n}$ . The embeddings are given by the singular vectors described in Theorem 3.8. The submodule  $\mathcal{C}'$  is embedded by the singular vector

$$|T\rangle^- = \mathcal{E}^{-, -n}(r, s, t) \mathcal{G}_{-n} \dots \mathcal{G}_{-1} |h_{\text{cm}}^-(n, r, s, t), l_{\text{cm}}^-(n, r, s, t), t\rangle. \quad (3.18)$$

Obviously,  $|T\rangle^-$  is inside the submodule generated from the charged singular vector

$$|E(n, h_{\text{cm}}^-(n, r, s, t), t)\rangle_{\text{ch}} = \mathcal{G}_{-n} \dots \mathcal{G}_{-1} |h_{\text{cm}}^-(n, r, s, t), l_{\text{cm}}^-(n, r, s, t), t\rangle. \quad (3.19)$$

Further, the top-level representative

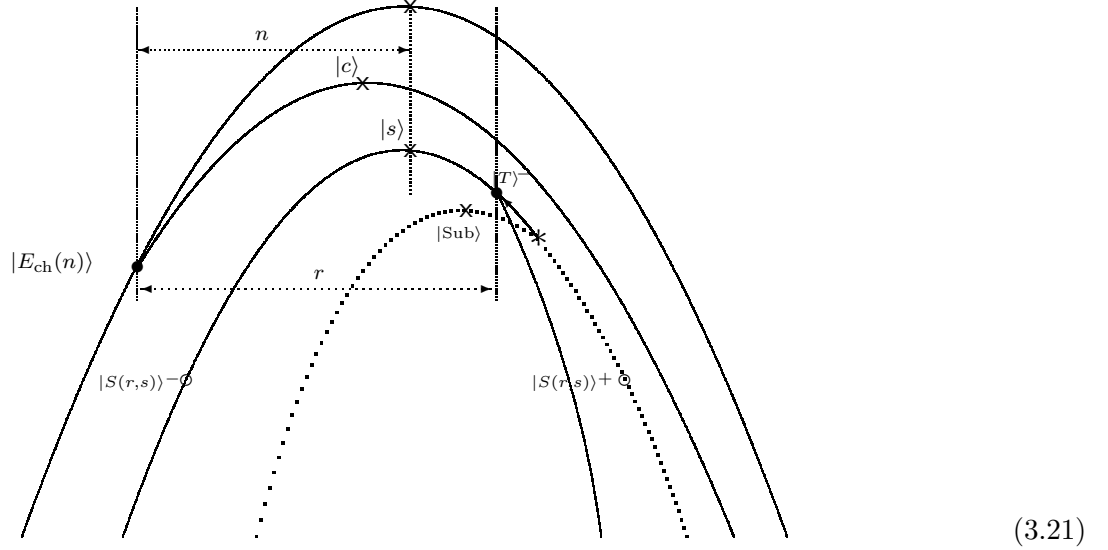
$$|c\rangle = \mathcal{Q}_1 \dots \mathcal{Q}_{n-1} \mathcal{G}_{-n} \dots \mathcal{G}_{-1} |h_{\text{cm}}^-(n, r, s, t), l_{\text{cm}}^-(n, r, s, t), t\rangle$$

of this charged singular vector generates the module  $\mathcal{C}$ . In the conventional description, the existence of a subsingular vector in the module  $\mathcal{U}_{h,\ell,t}$  depends on whether the top-level representative  $|s\rangle$  of the extremal diagram connecting  $|T\rangle^-$  and  $|S(r, s, h_{\text{cm}}^-(n, r, s, t), t)\rangle^+$  belongs to the submodule  $\mathcal{C}'$ . It is clear that

$$\begin{aligned} |s\rangle = \mathcal{G}_0 \dots \mathcal{G}_{r-n-1} |T\rangle^-, \quad r \geq n > 0 &\implies |s\rangle \in \mathcal{C}', \\ |T\rangle^- = \mathcal{G}_{r-n} \dots \mathcal{G}_{-1} |s\rangle, \quad n \geq r > 0 &\implies |s\rangle \notin \mathcal{C}', \end{aligned} \quad (3.20)$$

whence we see that the quotient of  $\mathcal{U}_{h,\ell,t}$  over conventional singular vectors is reducible in the case where  $r \geq n > 0$ .  $\square$

In terms of extremal diagrams, the conditions relating  $n$  and  $r$  mean that the twisted topological highest-weight state  $|T\rangle^-$  in (3.14) has gone past the top of the ‘massive’ parabola (i.e., past the conventional singular vector). Therefore, the extremal diagram actually takes the following form:



Here, the crosses denote the conventional, top-level, representatives, the  $\bullet$  states satisfy twisted topological highest-weight conditions,  $*$  is the state (a descendant of  $|S(r, s, h_{\text{cm}}^-(n, r, s, t), t)\rangle^+$ ) such that  $|T\rangle^- = \mathcal{G}_{-n+r} \cdot (*)$ , and  $|E(n)\rangle_{\text{ch}} \equiv |E(n, h_{\text{cm}}^-(n, r, s, t), t)\rangle_{\text{ch}}$  (and, as before, we consider the case where  $r \geq n > 0$ ). The arrow in the diagram, which represents the action of  $\mathcal{G}_{r-n}$ , cannot be inverted because of the twisted topological highest-weight conditions at  $|T\rangle^-$ , therefore the dotted line cannot be reached by the action of elements of the  $N=2$  algebra on either  $|T\rangle^-$  or the top-level representative  $|s\rangle$  (nor, in fact,  $|c\rangle$ ). Instead, acting with the highest of modes of  $\mathcal{Q}$  that produces a non-vanishing result, one spans out the lower branch originating at  $|T\rangle^-$ , which is shown in the solid line. After taking the quotient with respect to the singular vector  $|S(r, s)\rangle^- \equiv |S(r, s, h_{\text{cm}}^-(n, r, s, t), t)\rangle^-$  (or, *equivalently*,  $|s\rangle$ ), we are left with the submodule whose extremal diagram is precisely the dotted line. Then the state  $|\text{Sub}\rangle$  (the top-level representative of this diagram) is a *subsingular* vector.

However, rather than describing the structure of  $N=2$  Verma modules in terms of subsingular vectors, it is much more convenient to construct those vectors that do generate maximal submodules. In (3.21), this is the canonical representative  $|S(r, s)\rangle^+ \equiv |S(r, s, h_{\text{cm}}^-(n, r, s, t), t)\rangle^+$  from (2.45).

### 3.3 Codimension-3 cases.

Now we are going to analyze codimension-3 degenerations.

Let us begin with the case when a further degeneration occurs in Theorem 3.4, as described in case 7 of the list on page 15. Namely, one more massive Verma submodule appears in the diagram of the type of (3.14), with its own topological point similar to  $|T\rangle^-$ . All such topological points are at the same time topological singular vectors in the submodule generated from a charged singular vector (Lemma 3.3). In this way, the structure of submodules in the massive Verma module is still essentially described by that of its topological Verma submodule generated from a charged singular vector (while the structure of the topological Verma module, in turn, is the same as for the corresponding  $\widehat{sl}(2)$  Verma module).

**Theorem 3.10** *The highest-weight of the massive Verma module  $\mathcal{U}_{h,\ell,t}$  belongs to the set  $\mathcal{O}_{\text{cmm}}$  if and only if  $h = h_{\text{cm}}^\sigma(r, s, n, t)$ ,  $\ell = l_{\text{cm}}^\sigma(r, s, n, t)$ , where  $\sigma \in \{-, +\}$  and*

$$(\sigma, r, s, n, t) \in \left( (\{\pm\} \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z} \times \mathbb{Q}) \cup \mathbb{A}' \right) \setminus \mathbb{B}, \quad (3.22)$$

where  $\mathbb{B}$  is as in (3.12),

$$\mathbb{A}' = \left\{ (\{+\}, r, 0, n, t) \mid r \in \mathbb{N}, n \in \mathbb{N}, t \in \mathbb{Q} \right\} \cup \left\{ (\{-\}, r, 0, n, t) \mid r \in \mathbb{N}, n \in -\mathbb{N}_0, t \in \mathbb{Q} \right\},$$

and, with  $t = \frac{p}{q}$ ,

$$\left| r - \frac{ps}{q} \right| \notin \{|n|, |n| + 1, |n| + 2, \dots\}. \quad (3.23)$$

Then, the structure of  $\mathcal{U}_{h,\ell,t}$  is described as follows:

1. there exists a twisted topological Verma submodule  $\mathcal{C} = \mathfrak{V}_{h^\pm(r, s, \frac{p}{q}), \frac{p}{q}; -n} \hookrightarrow \mathcal{U}_{h_{\text{cm}}^\pm(r, s, n, \frac{p}{q}), l_{\text{cm}}^\pm(r, s, n, \frac{p}{q}), \frac{p}{q}}$ , where the embedding is given by singular vector (2.40);
2. any other primitive submodule in  $\mathcal{U}_{h_{\text{cm}}^\pm(r, s, n, \frac{p}{q}), l_{\text{cm}}^\pm(r, s, n, \frac{p}{q}), \frac{p}{q}}$  satisfies one of the following:
  - (a) it is a submodule in  $\mathcal{C}$  (hence, a twisted topological Verma module);
  - (b) it is a massive Verma module  $\mathcal{U}'$  generated from the representative  $\left| S(r, s, h_{\text{cm}}^\pm(r, s, n, \frac{p}{q}), \frac{p}{q}) \right\rangle^+$  (if  $n \geq 1$ ) or  $\left| S(r, s, h_{\text{cm}}^\pm(r, s, n, \frac{p}{q}), \frac{p}{q}) \right\rangle^-$  (if  $n \leq 0$ ) of the massive singular vector, where  $r, s \geq 1$ . Then there exists a vector  $\widetilde{|T\rangle}^\pm \in \mathcal{U}'$  that satisfies twisted massive highest-weight conditions with the twist parameter  $\theta = \mp r - n$  (if  $n \leq 0$ ) or  $\theta = \mp r - n + 1$  (if  $n > 0$ ) and such that the vector

$$|T\rangle^\pm = \begin{cases} \mathcal{Q}_{n \pm r} \widetilde{|T\rangle}^\pm = \mathcal{E}^{\pm, -n}(r, s + \frac{1}{2} \mp \frac{1}{2}, \frac{p}{q}) \mathcal{Q}_n \dots \mathcal{Q}_0 \left| h_{\text{cm}}^\pm(n, r, s, \frac{p}{q}), l_{\text{cm}}^\pm(n, r, s, \frac{p}{q}), \frac{p}{q} \right\rangle, & n \leq 0, \\ \mathcal{G}_{-n \mp r} \widetilde{|T\rangle}^\pm = \mathcal{E}^{\pm, -n}(r, s + \frac{1}{2} \pm \frac{1}{2}, \frac{p}{q}) \mathcal{G}_{-n} \dots \mathcal{G}_{-1} \left| h_{\text{cm}}^\pm(n, r, s, \frac{p}{q}), l_{\text{cm}}^\pm(n, r, s, \frac{p}{q}), \frac{p}{q} \right\rangle, & n \geq 1, \end{cases} \quad (3.24)$$

satisfies twisted topological highest-weight conditions and generates the twisted topological Verma module  $\mathcal{C}' = \mathcal{U}' \cap \mathcal{C}$ .

3. For any twisted topological Verma submodule  $\mathcal{C}' \subset \mathcal{C}$ , there exists a massive Verma submodule  $\mathcal{U}' \subseteq \mathcal{U}_{h,\ell,t}$  such that
  - (a)  $\mathcal{C}' \subset \mathcal{U}' \cap \mathcal{C}$  is a submodule in  $\mathcal{U}'$  corresponding to a charged singular vector;

- (b) there exists a vector  $|\widetilde{T}\rangle \in \mathcal{U}'$  that generates  $\mathcal{U}'$  such that the vector  $|T\rangle$  defined as in (3.24) generates a twisted topological Verma module that either coincides with  $\mathcal{C}'$  or contains  $\mathcal{C}'$  as a submodule generated from the topological singular vector  $|E(a, 1, \frac{p}{q})\rangle^{+, -n}$ ,  $a \in \mathbb{N}$ , or  $|E(a, 1, \frac{p}{q})\rangle^{-, -n}$ ,  $a \in \mathbb{N}$ , for  $n \geq 1$  and  $n \leq 0$  respectively.

In case 3 of the Theorem,  $\mathcal{U}' = \mathcal{U}_{h, \ell, t}$  occurs only when  $s = 0$  (when there are two charged singular vectors on one side of the highest-weight state), otherwise  $\mathcal{U}' \subsetneq \mathcal{U}_{h, \ell, t}$ . For negative rational  $t$  and small  $r$  and  $s$ , the excluded cases where only one massive Verma submodule exists are those covered by Theorem 3.8. As in the above,  $\mathcal{E}^{\pm, \theta}(r, s, t)$  denote topological singular vector operators transformed by the spectral flow with the parameter  $\theta$ .

PROOF. From the definition of  $\mathcal{O}_{\text{cmm}}$ , follow the equations on  $h$ ,  $\ell$ , and  $t$  with the solutions (3.22)–(3.23). Item 1 of the Theorem is a part of the definition of  $\mathcal{O}_{\text{cmm}}$ . Further, by Lemma 3.3, each twisted topological Verma submodule is embedded into the module  $\mathcal{C}$  by a topological singular vector. Each massive submodule has a non-empty intersection with  $\mathcal{C}$ . This intersection is generated from the topological singular vector written on the RHS of (3.24).

The crucial point in the Theorem is the existence of the  $|\widetilde{T}\rangle^{\pm}$  states. For definiteness, we choose  $n > 0$  and the ‘ $-$ ’ case in (3.24). We then apply the  $g$  operator of length  $-1$  to the twisted topological highest-weight state  $|T(r, s, n, t)\rangle \equiv |T\rangle^-$  from (3.24),

$$\begin{aligned} |\widetilde{T}\rangle^- &\equiv |\widetilde{T}(r, s, n, t)\rangle = g(r - n + 1, r - n - 1) |T(r, s, n, t)\rangle \\ &= g(r - n + 1, r - n - 1) \mathcal{E}^{-, -n}(r, s, t) |E(n, \mathbf{h}_{\text{cm}}^-(r, s, n, t), t)_{\text{ch}}, \end{aligned} \quad (3.25)$$

in accordance with the rules of Sec 2.2. The condition for the  $|\widetilde{T}(r, s, n, t)\rangle$  state to exist in the module  $\mathcal{U}_{\mathbf{h}_{\text{cm}}^-(n, r, s, t), \mathbf{l}_{\text{cm}}^-(n, r, s, t), t}$  is given by the next Lemma, from which we see that, under the conditions of the Theorem,  $f(r, s, n, t) \neq 0$  and therefore the  $|\widetilde{T}\rangle$  state does exist.  $\square$

**Lemma 3.11** *The state  $|\widetilde{T}(r, s, n, t)\rangle$ , Eq. (3.25), exists in the massive Verma module  $\mathcal{U}_{\mathbf{h}_{\text{cm}}^-(r, s, n, t), \mathbf{l}_{\text{cm}}^-(r, s, n, t), t}$  with  $1 \leq n \leq r(s + 1)$  if and only if  $f(r, s, n, t) \neq 0$ , where*

$$f(r, s, n, t) = \begin{cases} \prod_{i=0}^{2r-n} (st + n - r + i), & n \leq 2r, \\ 1, & n \geq 2r + 1. \end{cases} \quad (3.26)$$

*This state is then a representative of the massive singular vector; further, the dense  $\mathcal{Q}$ -descendant of  $|S(r, s, \mathbf{h}_{\text{cm}}^-(r, s, n, t), t)\rangle^+$  that lies in the same grade as (3.25) is proportional to that vector:*

$$\mathcal{G}_{r-n+1} \dots \mathcal{G}_{rs-1} |S(r, s, \mathbf{h}_{\text{cm}}^-(r, s, n, t), t)\rangle^+ = a(r, s, n, t) |\widetilde{T}(r, s, n, t)\rangle, \quad (3.27)$$

where  $a(r, s, n, t)$  is  $\left(\frac{2}{t}\right)^{rs}$  times a polynomial of the order  $r(s + 1)$  in  $t$ .

PROOF. To evaluate (3.25), one uses formulae (2.25) and then, as negative-length  $g$  operators reach the highest-weight state, one applies the formula,

$$g(\theta_1, \theta - 1) |h, \ell, t; \theta\rangle = \frac{1}{2\widehat{\ell}(h, \ell, t, \theta - \theta_1 + 1)} \mathcal{Q}_{-\theta_1+1} g(\theta_1 - 1, \theta - 1) |h, \ell, t; \theta\rangle, \quad (3.28)$$

(which also follows from Sec. 2.2) with

$$\widehat{\ell}(h, \ell, t, N) = \ell - \mathbf{l}_{\text{ch}}(N, h, t). \quad (3.29)$$

In the case at hand, we further use the fact that

$$\widehat{\ell}(\mathbf{h}_{\text{cm}}^-(r, s, n, t), \mathbf{l}_{\text{cm}}^-(r, s, n, t), t, -i) = -\frac{1}{t}(i + n)(st + n - r + i). \quad (3.30)$$

A simple analysis of the relative charge of  $|\widehat{S}^-(r, s, \mathbf{h}_{\text{cm}}^-(r, s, n, t), t)\rangle$  shows that  $f(r, s, n, t)$  is precisely the function responsible for the existence of  $|\widehat{T}(r, s, n, t)\rangle$  because, for  $t \neq 0$ , the relevant factors from the denominators are precisely the above  $f(r, s, n, t)$ , whence the lemma follows.  $\square$

While this case is rather straightforward when described in terms of singular vectors that generate maximal submodules, the analysis of the same structure in terms of top-level singular vectors and subsingular vectors that become necessary then is quite lengthy when it comes to listing all possible occurrences of subsingular vectors. Comparing (3.14) and (3.21) we have seen that the appearance of subsingular vectors in the conventional setting is due to the fact that the twisted topological highest-weight state is shifted to a certain side (depending on the sign, etc., of the parameters) of the top-level vector in the extremal diagram of the submodule. In the present case, however, there are two independent massive subdiagrams in the extremal diagram, each with its own ‘topological point’. The description in terms of conventional singular vectors and subsingular vectors would then amount to classifying all possible relative positions of the topological points and top-level vectors of the parabolas. Although this presents no conceptual difficulties and can be carried out similarly to Proposition 3.9, yet there are a large number of different cases. We omit this analysis, since it does not add anything to Theorem 3.10 as regards the structure of submodules, while at the same time is too long to serve as an example.

It remains to consider the case where, in addition to the conditions of Theorem 3.6, the submodules generated from the charged singular vector, in their own turn, admit topological singular vectors. Then, the corresponding twisted topological Verma submodules may be such that a given massive Verma submodule may be embedded into a direct sum of two such twisted topological Verma submodules. The corresponding massive singular vector then ‘splits’ into a pair of singular vectors, *each of which belongs to the respective twisted topological Verma submodule*. In the restricted setting with only top-level representatives allowed to generate submodules, this can be observed as the occurrence of two linearly independent singular vectors in the same grade<sup>10</sup> or as the appearance of a singular vector and a subsingular vector in the same grade.

As in Theorem 3.6, we assume for definiteness that the integers labelling the charged singular vectors in the massive Verma module  $\mathcal{U}_{h,\ell,t}$  are such that  $n > 0$  and  $m \leq 0$ . By the *distance* between any two vectors on the same extremal diagram we mean the difference of their  $U(1)$  charges. Then the distance between the charged singular vectors in  $\mathcal{U}_{\mathbf{h}_{\text{cc}}(n,m,t), \mathbf{l}_{\text{cc}}(n,m,t), t}$  is equal to  $-m + n + 1$ .

Now we are ready to describe case 8 of the list on page 15, i.e., the coexistence of a massive singular vector with two charged singular vectors on different sides of the highest-weight vector:

**Lemma 3.12** *The highest-weight of the massive Verma module  $\mathcal{U}_{h,\ell,t}$  belongs to the set  $\mathcal{O}_{\text{ccm}}$  if and only if*

$$h = \mathbf{h}_{\text{cc}}(n, m, t), \quad \ell = \mathbf{l}_{\text{cc}}(n, m, t), \quad t = \frac{r \pm (n-m)}{s}, \quad r, s, n \in \mathbb{N}, \quad m \in -\mathbb{N}_0. \quad (3.31)$$

*Then, the massive Verma module  $\mathcal{U}_{h,\ell,t}$  contains twisted topological submodules  $\mathcal{C}_1$  and  $\mathcal{C}_2$  generated from the charged singular vectors  $|E(n, \frac{(1-m-n)s}{r \mp (m-n)}, \frac{r \pm (n-m)}{s})\rangle_{\text{ch}}$  and  $|E(m, \frac{(1-m-n)s}{r \mp (m-n)}, \frac{r \pm (n-m)}{s})\rangle_{\text{ch}}$  respectively. Each of the modules  $\mathcal{C}_1$  and  $\mathcal{C}_2$  admits a singular vector; moreover, a singular vector  $|E(a, b, t)\rangle^{\mp, -n}$  exists in  $\mathcal{C}_1$  if and only if  $|E(a, b, t)\rangle^{\pm, -m}$  exists in  $\mathcal{C}_2$  with the same  $a, b \in \mathbb{N}$  (and with the above  $t$ ). Any other primitive submodule  $\mathcal{U}' \subset \mathcal{U}_{h,\ell,t}$  satisfies one of the following:*

<sup>10</sup>The occurrence of linearly independent singular vectors in the same grade was noticed for the first time in [D]. As we are going to see, they are necessarily elements of twisted topological, not massive, Verma submodules.

1. it is a twisted topological Verma module, in which case it is a submodule of either  $\mathcal{C}_1$  or  $\mathcal{C}_2$ ;
2. it is a massive Verma module, in which case the non-empty intersections  $\mathcal{C}'_1 = \mathcal{U}' \cap \mathcal{C}_1$  and  $\mathcal{C}'_2 = \mathcal{U}' \cap \mathcal{C}_2$  are generated each from a topological singular vector in  $\mathcal{C}_1$  and  $\mathcal{C}_2$  respectively. If, then,  $\mathcal{C}'_1$  is generated from the singular vector  $|E(a, b, t)\rangle^{\pm, -n}$ , then  $\mathcal{C}'_2$  is generated from the singular vector  $|E(a, b, t)\rangle^{\mp, -m}$  with the same  $a, b \in \mathbb{N}$  (and with the above  $t$ ). Each of the submodules  $\mathcal{C}'_1$  and  $\mathcal{C}'_2$  is at the same time generated from a charged singular vector in  $\mathcal{U}'$ .

PROOF. The definition of  $\mathcal{O}_{\text{ccm}}$  means that both solutions of Eq. (2.37) are integers of different signs and, in addition, the states (2.38) admit a topological singular vector, whence (3.12) follows. Further, each of the modules  $\mathcal{C}_1$  and  $\mathcal{C}_2$  generated from the charged singular vectors contains a topological singular vector. Now assuming that the highest-weight vector of  $\mathcal{C}_1$  is  $|h^{\pm}(a, b, t), t; -n\rangle_{\text{top}}$ , we see that the highest-weight vector of  $\mathcal{C}_2$  is  $|h^{\mp}(a, b, t), t; -m\rangle_{\text{top}}$ . Therefore the assertion that a singular vector  $|E(a, b, t)\rangle^{\mp, -n}$  exists in  $\mathcal{C}_1$  if and only if  $|E(a, b, t)\rangle^{\pm, -m}$  exists in  $\mathcal{C}_2$  with the same  $a, b \in \mathbb{N}$  is obvious.

Note further that the quotient of  $\mathcal{U}_{h, \ell, t}$  over  $\mathcal{C}_1$  or  $\mathcal{C}_2$  is a twisted topological Verma module  $\mathcal{Q}_1$  or  $\mathcal{Q}_2$  respectively. Let us assume that there exists a twisted topological highest-weight state  $|t\rangle$  such that  $|t\rangle \notin \mathcal{C}_1$ ,  $|t\rangle \notin \mathcal{C}_2$ . Then this state is a topological singular vector in  $\mathcal{Q}_1$  and, likewise, in  $\mathcal{Q}_2$ . However, the Verma module generated from  $|t\rangle$  contains states in the gradings where there are no states from either the  $\mathcal{Q}_1$  or the  $\mathcal{Q}_2$  module. Therefore, the state  $|t\rangle$  belongs to a twisted topological Verma module and, at the same time, generates a submodule which is isomorphic to the quotient of a twisted topological Verma module. This contradicts the structure of the topological Verma modules described in Theorem 3.1. Part 2 follows immediately from Lemma 3.3.  $\square$

In the case described in the Lemma, therefore, a given massive Verma submodule  $\mathcal{U}'$  in  $\mathcal{U}_{h, \ell, t}$  necessarily has two charged singular vectors lying on the different sides of the highest-weight vector of  $\mathcal{U}'$ . It may be useful to recall the diagram (3.14), where the topological singular vector  $|T\rangle^-$  is, at the same time, a charged singular vector in the massive Verma module whose extremal diagram is the parabola connecting  $|S(r, s)\rangle^-$  and  $|S(r, s)\rangle^+$ . In the present case, we have two topological points on the extremal diagram of any massive submodule, which are the highest-weight states of the modules  $\mathcal{C}'_1$  and  $\mathcal{C}'_2$ :

$$\begin{array}{ccccc}
 & & \mathcal{U} & & \\
 & \nearrow & \uparrow & \nwarrow & \\
 \mathcal{C}_1 & & \mathcal{U}' & & \mathcal{C}_2 \\
 & \nwarrow & \downarrow & \nearrow & \\
 & \mathcal{C}'_1 & & \mathcal{C}'_2 & 
 \end{array} \tag{3.32}$$

Conversely, let us be given any topological singular vector in  $\mathcal{C}_1$ ,

$$|e_1\rangle = \mathcal{E}^{\pm, -n}(a, b, t) \mathcal{G}_{-n} \dots \mathcal{G}_{-1} |h_{\text{cc}}(n, m, t), l_{\text{cc}}(n, m, t), t\rangle, \tag{3.33}$$

(with  $t$  as in the Lemma). Then we find the corresponding singular vector in  $\mathcal{C}_2$ :

$$|e_2\rangle = \mathcal{E}^{\mp, -m}(a, b, t) \mathcal{Q}_m \dots \mathcal{Q}_0 |h_{\text{cc}}(n, m, t), l_{\text{cc}}(n, m, t), t\rangle. \tag{3.34}$$

Now the question is whether a massive submodule  $\mathcal{U}'$  exists in the Verma module under consideration such that (3.33) and (3.34) would be charged singular vectors in that massive Verma submodule. This is not always the case. In more detail, the structure of the module  $\mathcal{U}$  is described in the following

**Theorem 3.13** *Under the conditions of Lemma 3.12,*

1. *Whenever  $t = \frac{-m+n+r}{s}$ , the module  $\mathcal{C}_1$  contains the state*

$$|e_1\rangle = \mathcal{E}^{+,-n}(r, s+1, t) \mathcal{G}_{-n} \dots \mathcal{G}_{-1} |h_{cc}(n, m, t), l_{cc}(n, m, t), t\rangle, \quad (3.35)$$

*that satisfies twisted topological highest-weight conditions. Let then*

$$|e_2\rangle = \mathcal{E}^{-,-m}(r, s+1, t) \mathcal{Q}_m \dots \mathcal{Q}_0 |h_{cc}(n, m, t), l_{cc}(n, m, t), t\rangle \quad (3.36)$$

*be the singular vector in  $\mathcal{C}_2$  whose existence is claimed in the Lemma. There exist states  $\widetilde{|T\rangle}^-$  and  $\widetilde{|T\rangle}^+$  in  $\mathcal{U}_{h_{cc}(n, m, t), l_{cc}(n, m, t), t}$  such that*

$$\begin{aligned} \mathcal{G}_{-r-n} \widetilde{|T\rangle}^+ &= |e_1\rangle, \\ \mathcal{Q}_{-r+m} \widetilde{|T\rangle}^- &= |e_2\rangle. \end{aligned} \quad (3.37)$$

*Each of these two states generates the same massive Verma submodule  $\mathcal{U}' \subset \mathcal{U}_{h, \ell, t}$ , in which  $|e_1\rangle$  and  $|e_2\rangle$  are charged singular vectors.*

2. *Whenever  $t = \frac{m-n+r}{s}$ , the module  $\mathcal{C}_1$  contains the state*

$$|e_1\rangle = \mathcal{E}^{-,-n}(r, s, t) \mathcal{G}_{-n} \dots \mathcal{G}_{-1} |h_{cc}(n, m, t), l_{cc}(n, m, t), t\rangle \quad (3.38)$$

*that satisfies twisted topological highest-weight conditions. Let then  $|e_2\rangle$  be a singular vector in  $\mathcal{C}_2$  whose existence is claimed in the Lemma. Then,*

- (a) *if  $2r + m - n \leq -1$ , there exists a massive Verma submodule  $\mathcal{U}' \subset \mathcal{U}_{h, \ell, t}$  generated from any of the states  $\widetilde{|v_1\rangle}$  or  $\widetilde{|v_2\rangle}$  such that*

$$\begin{aligned} \mathcal{G}_{r-n} \widetilde{|v_1\rangle} &= |e_1\rangle, \\ \mathcal{Q}_{r+m} \widetilde{|v_2\rangle} &= |e_2\rangle, \end{aligned} \quad (3.39)$$

*and, further,  $|e_1\rangle$  and  $|e_2\rangle$  are charged singular vectors in  $\mathcal{U}'$ , the distance between them being  $-m + n + 1 - 2r$ ;*

- (b) *if  $2r + m - n \geq 0$ , there does not exist a massive submodule  $\mathcal{U}'$  in  $\mathcal{U}_{h, \ell, t}$  in which either  $|e_1\rangle$  or  $|e_2\rangle$  would be a charged singular vector. If, further,  $2r + m - n \geq 1$ , then,*

- *for each  $i$  from the range  $i = 0, \dots, 2r + m - n - 1$ , the states  $\mathcal{Q}_{r+m-i} \dots \mathcal{Q}_{r+m-1} |e_2\rangle$  and  $\mathcal{G}_{-m-r+i+1} \dots \mathcal{G}_{r-n-1} |e_1\rangle$  satisfy twisted massive highest-weight conditions, are in the same grade and are linearly independent;*
- *the modules  $\mathcal{C}'_1$  and  $\mathcal{C}'_2$  generated by  $|e_1\rangle$  and  $|e_2\rangle$  respectively, contain topological singular vectors  $\mathcal{G}_{-r-m} \dots \mathcal{G}_{r-n-1} |e_1\rangle$  and  $\mathcal{Q}_{-r+n} \dots \mathcal{Q}_{r+m-1} |e_2\rangle$ , respectively.*

PROOF. In case 1 of the Theorem, the states  $\widetilde{|T\rangle}^-$  and  $\widetilde{|T\rangle}^+$  can be written as

$$\begin{aligned} \widetilde{|T\rangle}^+ &= g(-r - n + 1, -r - n - 1) |e_1\rangle, \\ \widetilde{|T\rangle}^- &= q(-r + m + 1, -r + m - 1) |e_2\rangle. \end{aligned} \quad (3.40)$$

They exist as elements of  $\mathcal{U}_{h, \ell, t}$  in view of the argument similar to the one used in Lemma 3.11. The fact that  $\widetilde{|T\rangle}^-$  and  $\widetilde{|T\rangle}^+$  are dense  $\mathcal{G}/\mathcal{Q}$ -descendants of each other (up to a nonzero factor) is checked by

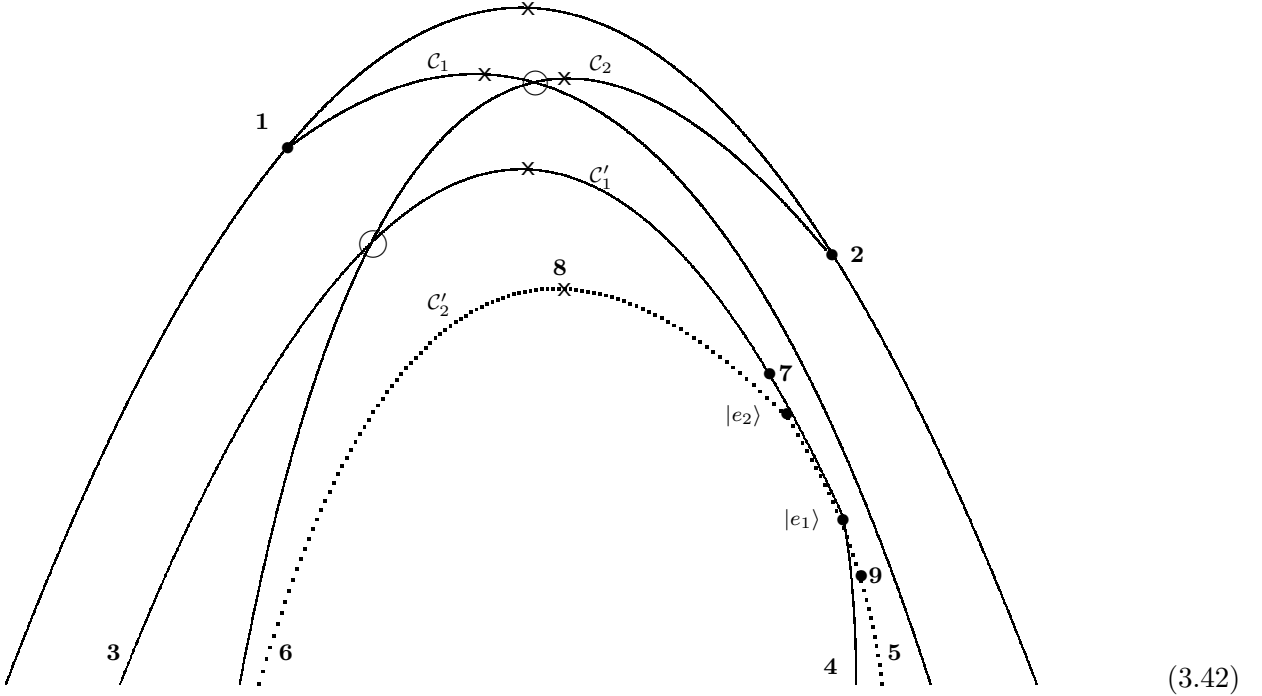
quotienting  $\mathcal{U}_{\text{hcc}(n,m,t),\text{lcc}(n,m,t),t}$  over  $\mathcal{C}_1$  or  $\mathcal{C}_2$ . The assumption that dense  $\mathcal{G}/\mathcal{Q}$ -descendants of  $|\widetilde{T}\rangle^-$  and of  $|\widetilde{T}\rangle^+$  in the same grading are linearly independent leads to the contradiction with the structure of the quotient  $\mathcal{U}_{\text{hcc}(n,m,t),\text{lcc}(n,m,t),t}/\mathcal{C}_1$ , which is a twisted topological Verma module.

In case 2, Lemma 3.11 assures that the states

$$\begin{aligned} |\widetilde{v}_1\rangle &= g(r-n+1, r-n-1)|e_1\rangle, \\ |\widetilde{v}_2\rangle &= q(r+m+1, r+m-1)|e_2\rangle \end{aligned} \quad (3.41)$$

exist in case (a) and do not exist in  $\mathcal{U}_{h,\ell,t}$  in case (b).  $\square$

Different cases in the Theorem are thus distinguished by whether or not there exists a *massive* Verma submodule  $\mathcal{U}'$  such that its intersections with  $\mathcal{C}_1$  and  $\mathcal{C}_2$  coincide with submodules  $\mathcal{C}'_1 \subset \mathcal{C}_1$  and  $\mathcal{C}'_2 \subset \mathcal{C}_2$  generated from the singular vectors  $|e_1\rangle$  and  $|e_2\rangle$  respectively. In cases 1 and 2a, one has the embeddings as in (3.32), whereas in case 2b there is no massive Verma submodule  $\mathcal{U}'$  in which the highest-weight vectors of  $\mathcal{C}'_1$  and  $\mathcal{C}'_2$  would be charged singular vectors. Case 2b of the Theorem can be illustrated as

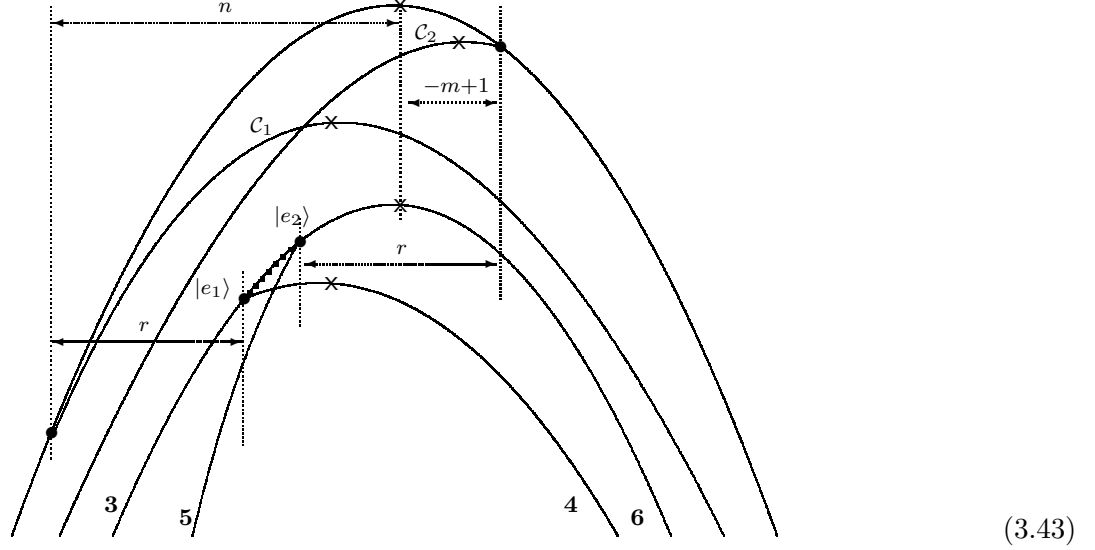


Here, **1** and **2** are the charged singular vectors in the massive Verma module  $\mathcal{U}$ , which read  $\mathcal{G}_{-n} \dots \mathcal{G}_{-1} \cdot | \frac{(1-m-n)s}{m-n+r}, \frac{mns}{n-m-r}, \frac{m-n+r}{s} \rangle$  and  $\mathcal{Q}_m \dots \mathcal{Q}_0 \cdot | \frac{(1-m-n)s}{m-n+r}, \frac{mns}{n-m-r}, \frac{m-n+r}{s} \rangle$ , respectively. The extremal diagrams of the corresponding twisted topological submodules in  $\mathcal{U}$  are labelled by  $\mathcal{C}_1$  and  $\mathcal{C}_2$  respectively. The top-level representatives (2.41) are marked with crosses. The extremal diagrams of the twisted topological Verma submodules generated from  $|e_1\rangle$  and  $|e_2\rangle$  respectively are given by **3**– $|e_1\rangle$ –**4**, with the cusp at  $|e_1\rangle$ , and by **5**– $|e_2\rangle$ –**6**, with the cusp at  $|e_2\rangle$ . Thus, neither  $|e_1\rangle$  nor  $|e_2\rangle$  alone generates all of the states in **3**– $|e_2\rangle$ – $|e_1\rangle$ –**5**. Those states of the two submodules that lie between  $|e_1\rangle$  and  $|e_2\rangle$  are in the same grade and are linearly independent. They satisfy twisted massive highest-weight conditions and might thus be taken for two linearly independent massive singular vectors in the same grade (when, e.g., such  $|e_1\rangle$  and  $|e_2\rangle$  happen to lie on different sides of the top of the parabola, one would observe the pair of conventional

singular vectors [D] at the top of the parabola). However, we have seen that each of the two linearly independent states in the same grade belongs in fact to its own twisted *topological* Verma submodule.

The state at  $\mathbf{7} \in \mathcal{C}'_1$  is yet another topological singular vector from part 2b of the Theorem, in particular  $\mathcal{Q}_{r+m} \mathbf{7} = 0$ . The state  $|s_1\rangle \in \mathcal{C}'_1$  (not indicated in the diagram), which is in the same grade as  $|e_2\rangle \in \mathcal{C}'_2$  but belongs to the other twisted topological submodule, is such that  $\mathcal{G}_{-r-m} |s_1\rangle = \mathbf{7}$ . (Similarly,  $\mathbf{9} \in \mathcal{C}'_2$  is a topological singular vector as well). The state  $\mathbf{8}$  is the top-level representative of the extremal diagram generated by the topological singular vector  $|e_2\rangle$ , but is not in the module generated from  $|e_1\rangle$ .

Similarly, in case 2a of Theorem 3.13, we have the extremal diagram



where  $\mathbf{3}-|e_1\rangle-\mathbf{4}$  is the extremal diagram of the twisted topological Verma submodule  $\mathcal{C}'_1 = \mathcal{C}_1 \cap \mathcal{U}'$ , and  $\mathbf{5}-|e_2\rangle-\mathbf{6}$ , that of  $\mathcal{C}'_2 = \mathcal{C}_2 \cap \mathcal{U}'$ . There are  $-m + n - 2r$  states between  $|e_1\rangle$  and  $|e_2\rangle$  that satisfy twisted massive highest-weight conditions but *do not belong to either  $\mathcal{C}'_1$  or  $\mathcal{C}'_2$  submodules*, nor, in fact, to *either the  $\mathcal{C}_1$  or  $\mathcal{C}_2$  submodules of  $\mathcal{U}_{h,\ell,t}$* . Thus, these states survive in the quotient module with respect to  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . It is these states that generate the entire massive submodule  $\mathbf{3}-|e_1\rangle-|e_2\rangle-\mathbf{6}$ , in which  $|e_1\rangle$  and  $|e_2\rangle$  are charged singular vectors.

As regards describing the above pictures in terms of only the top-level representatives of singular vectors and the subsingular vectors, one has to consider submodules of the submodules described above generated by the conventional, top-level, representatives of the extremal diagrams. The missing parts of the submodules will then be generated by subsingular vectors. For the ‘traditional’ reasons, we now briefly describe the subsingular vectors ‘hidden’ in the above pictures.

To begin with case 2b, consider what happens in (3.42) after factoring away the submodule generated from the top-level ( $\times$ ) representative of the extremal diagram  $\mathbf{3}-|e_1\rangle-\mathbf{4}$  of the  $\mathcal{C}'_1$  submodule. The vector  $|s_1\rangle$  (not shown in (3.42)) that lies at the same grade as  $|e_2\rangle \in \mathcal{C}'_2$  but belongs to  $\mathcal{C}'_1$  then satisfies twisted topological highest-weight conditions, thus giving rise to (the extremal diagram of) a twisted topological Verma submodule. The top-level representative of this extremal diagram is then a subsingular vector in the conventional sense. This top-level representative would be at the same point as  $\mathbf{8}$ , the top-level representative of the topological singular vector  $|e_2\rangle$  in  $\mathcal{U}$ .

Thus, the top-level representative of  $|s_1\rangle$  does not belong to the submodule generated by the top-level representative of  $\mathcal{C}'_1$  because of the topological highest-weight conditions at  $\mathbf{7}$ . We also see that this

subsingular vector lies in the same grade as **8**. Thus – continuing with the conventional definition of singular vectors – a singular vector and a subsingular vector are in the same grade in the case under consideration. The crucial feature of this case is that the entire sections **7**– $|e_1\rangle$  and  $|e_2\rangle$ –**9** of the extremal diagrams of each of the topological submodules are on one side of the top of the parabola. Had these sections included the top-level representative, one would conclude that two conventional singular vectors exist in the same grade. Whether or not this is the case is determined by parameters  $r$ ,  $s$ ,  $m$ , and  $n$ . As they change, one of these conventional singular vectors ‘submerges’ and becomes *subsingular*.

From the above discussion we immediately obtain the following three Propositions.

**Proposition 3.14** *Let the conditions of case 2b of Theorem 3.13 hold. Then, subsingular vectors exist in  $\mathcal{U} \equiv \mathcal{U}_{\frac{(1-m-n)s}{m-n+r}, \frac{mns}{n-m-r}, \frac{m-n+r}{s}}$  if and only if either  $r < -m + 1$  or  $r < n$ . In the first case, the subsingular vector reads*

$$\begin{aligned} |\text{Sub}\rangle &= \mathcal{G}_0 \dots \mathcal{G}_{-r-m-1} \cdot \mathcal{G}_{-r-m+1} \dots \mathcal{G}_{r-n-1} |e_1\rangle \\ &= \mathcal{G}_0 \dots \mathcal{G}_{-r-m-1} \cdot \mathcal{G}_{-r-m+1} \dots \mathcal{G}_{r-n-1} \mathcal{E}^{-, -n}(r, s, \frac{m-n+r}{s}) \mathcal{G}_{-n} \dots \mathcal{G}_{-1} \left| \frac{(1-m-n)s}{m-n+r}, \frac{mns}{n-m-r}, \frac{m-n+r}{s} \right\rangle \end{aligned} \quad (3.44)$$

(where  $\mathcal{E}^{-, \theta}(r, s, t)$  is the spectral flow transform of the topological singular vector operator read off from (2.34)). This becomes singular in the quotient module  $\mathcal{U}/\mathcal{C}_1''$ , where  $\mathcal{C}_1''$  is the submodule generated by the top-level representative of the extremal diagram **3**–**7**, which reads

$$\mathcal{G}_0 \dots \mathcal{G}_{r-n-1} \mathcal{E}^{-, -n}(r, s, \frac{m-n+r}{s}) \mathcal{G}_{-n} \dots \mathcal{G}_{-1} \left| \frac{(1-m-n)s}{m-n+r}, \frac{mns}{n-m-r}, \frac{m-n+r}{s} \right\rangle.$$

In the other case, the subsingular vector (in the conventional nomenclature) is given by an equally simple construction.

Similarly, describing case 2a of Theorem 3.13 in terms of only top-level representatives of singular vectors (and then, in terms of subsingular vectors) one would conclude that whenever  $r \geq |m| + 1$  or  $r \geq n$  the states between  $|e_1\rangle$  and  $|e_2\rangle$  in the diagram (3.43) are not generated from the top-level representatives. The top-level representative of the state  $|v_2\rangle$  (such that  $\mathcal{Q}_{r+m} \widetilde{|v_2\rangle} = |e_2\rangle$ ) gives a subsingular vector:

**Proposition 3.15** *Under the conditions of case 2a of Theorem 3.13, the state*

$$\begin{aligned} |\text{Sub}\rangle &= \mathcal{Q}_1 \dots \mathcal{Q}_{r+m-1} \widetilde{|v_2\rangle} \\ &= \mathcal{Q}_1 \dots \mathcal{Q}_{r+m-1} q(r+m+1, r+m-1) |e_2\rangle \\ &= \mathcal{Q}_1 \dots \mathcal{Q}_{r+m-1} q(r+m+1, r+m-1) \mathcal{E}^{+, -m}(r, s, \frac{n-m+r}{s}) \mathcal{Q}_m \dots \mathcal{Q}_0 \left| \frac{(1-m-n)s}{n-m+r}, \frac{mns}{m-n-r}, \frac{n-m+r}{s} \right\rangle \end{aligned} \quad (3.45)$$

is a subsingular vector in the module  $\mathcal{U} \equiv \mathcal{U}_{\frac{(1-m-n)s}{n-m+r}, \frac{mns}{m-n-r}, \frac{n-m+r}{s}}$  with  $r \geq |m| + 1$ . It becomes singular in the quotient module  $\mathcal{U}/(\mathcal{C}_1 \cup \mathcal{C}_2)$ , where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are submodules in  $\mathcal{U}$  generated by the charged singular vectors  $|E(n, \frac{(1-m-n)s}{n-m+r}, \frac{n-m+r}{s})\rangle_{\text{ch}}$  and  $|E(m, \frac{(1-m-n)s}{n-m+r}, \frac{n-m+r}{s})\rangle_{\text{ch}}$  respectively.

The length- $(-1)$  operator  $q(r+m+1, r+m-1)$  evaluates according to the rules given in Sec. 2.2. When  $r \geq n$ , the subsingular vector is built similarly starting from the vector  $\widetilde{|v_1\rangle} = g(r-n+1, r-n-1)|e_1\rangle$ .

Finally, even if one insists on considering only top-level representatives of singular vectors, there would be no subsingular vectors in case 1 of the Theorem, because topological states in the extremal diagram of the submodule have relative charges of different signs, hence the entire maximal submodule can be generated from the top-level representative.

**Proposition 3.16** *Under the conditions of case 1 of Theorem 3.13, no subsingular vectors exist in  $\mathcal{U}_{h, \ell, t}$ .*

## 4 Concluding remarks

We have analyzed the structure of  $N = 2$  Verma modules and classified their degeneration patterns. We considered singular vectors that generate maximal submodules (and satisfy twisted highest-weight conditions), which has allowed us to describe the structure of submodules of  $N = 2$  Verma modules in a setting which is free of subsingular vectors. However, in order to make contact with the approach existing in the literature, we have also shown how the description in terms of the conventional, ‘untwisted’, singular vectors and the subsingular vectors, as well as general expressions for the subsingular vectors, follow from our approach and the expressions for the singular vectors satisfying the twisted highest-weight conditions.

As we have seen, an important point about the structure of massive  $N = 2$  Verma modules is that there are submodules of exactly two different types, the massive and the twisted topological ones (and, obviously, arbitrary sums thereof). The existence of two types of submodules shows up also in the classification of the patterns describing possible sequences of submodules *of submodules* of a given  $N = 2$  Verma module, which we have not considered yet. This amounts to finding embedding diagrams of  $N = 2$  Verma modules. Using the singular vectors constructed in this paper, these would be *embedding* diagrams, i.e., those consisting only of mappings with trivial kernels. The sought sequence in which submodules may follow one another is determined by the degeneration patterns found in this paper. As regards the topological Verma modules, the answer is already known, since the corresponding embedding diagrams are isomorphic to those of  $\widehat{sl}(2)$  Verma modules. Further, we have seen that some of the degeneration patterns of massive  $N = 2$  Verma modules are such that, again, the structure of submodules is determined by that of a certain topological (hence,  $\widehat{sl}(2)$ -) Verma module. It thus remains to analyze several cases where the known embedding diagrams are ‘glued together’ to produce somewhat more complicated structures [SSi]. The classification of  $N = 2$  embedding diagrams is, thus, a refinement of the classification of degeneration patterns presented in this paper.

In view of the relation existing between  $\widehat{sl}(2|1)$  and  $N = 2$  singular vectors [S2], it would also be interesting to see how  $\widehat{sl}(2|1)$  subsingular vectors behave under the reduction [S2] to  $N = 2$  Verma modules.

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## Appendix: The proof of part iii) of Lemma 3.3

This proof exploits heavily the properties of extremal diagrams of both the twisted topological and the massive Verma modules (in fact, the reader would find the proof easier to read if he draws the parabolas, dense descendants, etc., which we deal with in what follows). The idea of the proof is to demonstrate that the converse leads to a contradiction either with the ‘size’ of twisted topological Verma modules (i.e., the appearance of states with bigradings outside the extremal diagram) or with the structure of submodules in twisted topological Verma modules (Theorem 3.1). This argument is applied several times to the topological Verma modules that are the quotients of the massive Verma module with respect to the topological Verma modules whose existence is established at previous steps of the proof.

To begin with, note that the module  $\mathcal{C}'$  cannot be embedded by charged singular vectors in more than one massive submodule in  $\mathcal{U}$ . Indeed, the assumption that  $\mathcal{U}' \supset \mathcal{C}' \subset \mathcal{U}''$ , where the embeddings are given by charged singular vectors, leads to the contradiction, because either  $\mathcal{U}'$  or  $\mathcal{U}''$  then necessarily has states in the gradings outside the module  $\mathcal{U}$ . To see this, let  $|v'\rangle$  be the highest-weight vector of  $\mathcal{C}'$ . Let  $|v'\rangle$  have the twist parameter  $\theta'$  and lie in the bigrading  $(\ell', h')$ . Then, the extremal states of the massive Verma submodule that contains  $|v'\rangle$  as one of its extremal states have to lie in bigradings  $(\ell, h)$ , all of which satisfy one and only one of the following equations:

$$\frac{1}{2}h^2 - (\frac{1}{2} + h' + \theta')h + \frac{1}{2}h' + \frac{1}{2}h'^2 + \ell' + h'\theta' = \ell, \quad (\text{A.1})$$

$$\frac{1}{2}h^2 + (\frac{1}{2} - h' - \theta')h - \frac{1}{2}h' + \frac{1}{2}h'^2 + \ell' + h'\theta' = \ell. \quad (\text{A.2})$$

Now, the following alternative is satisfied:

- either infinitely many bigradings satisfying (A.1) lie outside the module  $\mathcal{U}$ , in which case none of the bigradings satisfying (A.2) lie outside the module  $\mathcal{U}$ ,
- or infinitely many bigradings satisfying (A.2) lie outside the module  $\mathcal{U}$ , in which case none of the bigradings satisfying (A.1) lie outside the module  $\mathcal{U}$

(the converse would contradict the fact that  $\mathcal{U}$  is freely generated). Thus, we can associate with each state  $|v'\rangle$  two sets of bigradings  $(\ell_i^1, h_i^1)$  and  $(\ell_i^2, h_i^2)$ , each of which satisfies one and only one of Eqs. (A.1) and (A.2). Infinitely many bigradings from one set lie outside  $\mathcal{U}$ , while all bigradings from the other set lie inside  $\mathcal{U}$ . We will call bigradings from the latter set admissible with respect to the state  $|v'\rangle$ .

Thus, there may exist at most one massive submodule  $\mathcal{U}'$  into which  $\mathcal{C}'$  is embedded by a charged singular vector and, moreover, all of the extremal states of  $\mathcal{U}'$  have admissible bigradings with respect to the highest-weight vector of  $\mathcal{C}'$  if such a  $\mathcal{U}'$  exists. In the case where there *is* such a massive Verma module, we are in the situation described in Part ii) of the Lemma, using which iii) is proved. Consider, therefore, the case where

there does not exist a massive submodule  $\mathcal{U}' \supset \mathcal{C}'$  such that the embedding is given by a charged singular vector. (A.3)

It is easy to see then that there exists a state  $|y\rangle$  that satisfies the following properties (the proof of this statement is left to the reader as a useful exercise):

- $|y\rangle$  has an admissible bigrading with respect to  $|v'\rangle$ ;
- $|y\rangle$  satisfies twisted topological highest-weight conditions;
- unless  $|y\rangle = |v'\rangle$ , the vector  $|v'\rangle$  is a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of  $|y\rangle$ , while  $|y\rangle$  is not a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of  $|v'\rangle$ ;
- there are no states  $|z\rangle$  with admissible bigradings with respect to  $|y\rangle$  such that  $|y\rangle$  is a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of  $|z\rangle$ , while  $|z\rangle$  is not a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of  $|y\rangle$ .

It is clear that  $|y\rangle$  generates a twisted topological Verma module  $\mathcal{C}'' \supseteq \mathcal{C}'$  ( $\mathcal{C}'' = \mathcal{C}'$  whenever  $|y\rangle = |v'\rangle$ ) with the condition (A.3) satisfied for  $\mathcal{C}''$ . Now, we will have proved iii) for the module  $\mathcal{C}'$  as soon as we prove iii) for  $\mathcal{C}''$ . Let  $|y\rangle$  have the twist parameter  $\theta_y$  and the bigrading  $(\ell_y, h_y)$ . Assume, for definiteness, that any admissible bigrading with respect to  $|y\rangle$  satisfies (A.1) with  $\theta' = \theta_y$  and  $(\ell', h') = (\ell_y, h_y)$ . Then, the fact that there are no states  $|z\rangle$  with the properties as described above is equivalent to the fact that the expression  $g(\theta_y + 1, \theta_y - 1)|y\rangle$  cannot be evaluated as a polynomial in the modes of  $\mathcal{Q}$ ,  $\mathcal{G}$ ,  $\mathcal{L}$ , and  $\mathcal{H}$  acting

on the highest-weight vector of  $\mathcal{U}$ . By lengthy but direct calculations with formulae from Sec. 2.2 one can show that  $g(\theta_y + 1, \theta_y - 1)|y\rangle$  cannot be evaluated in this way if and only if there exists a twisted topological Verma module  $\mathcal{C}_1 \subset \mathcal{U}$ , where the embedding is given by a charged singular vector and such that  $\mathcal{C}_1$  is maximal ( $\mathcal{U} \supset \mathcal{C}'' \supset \mathcal{C} \implies \mathcal{C}'' = \mathcal{C}$ ) and  $\mathcal{C}_1 \cap \mathcal{C}'' = \{0\}$ . Then, consider the quotient  $\mathcal{Q}_1 = \mathcal{U}/\mathcal{C}_1$ . This is a twisted topological Verma module, which contains the topological singular vector  $|y\rangle$ . It follows by comparing the highest-weight parameters of  $\mathcal{Q}_1$  and  $\mathcal{C}_1$  that  $\mathcal{C}_1$  contains a topological singular vector  $|x\rangle$ . The bigrading of  $|x\rangle$  is  $(\ell_x, h_x) = (\ell_y + \theta_y, h_y - 1)$  and any bigrading admissible with respect to  $|y\rangle$  is admissible with respect to  $|x\rangle$ , and vice versa. Let  $\mathcal{C}'_1$  be the twisted topological Verma module generated from  $|x\rangle$ . We have two possibilities:

- a) there does not exist a massive submodule  $\mathcal{U}' \supset \mathcal{C}'_1$ , where the embedding is given by a charged singular vector;
- b) there exists a massive submodule  $\mathcal{U}' \supset \mathcal{C}'_1$ , where the embedding is given by a charged singular vector.

In case a), we can apply to  $\mathcal{C}'_1$  the same reasoning as in the case of the  $\mathcal{C}''$  module. In this way, we see that a module  $\mathcal{C}_2 \subset \mathcal{U}$  exists, where the embedding is given by a charged singular vector,  $\mathcal{C}_2$  is maximal ( $\mathcal{U} \supset \mathcal{C}''' \supset \mathcal{C}_2 \implies \mathcal{C}''' = \mathcal{C}_2$ ), and  $\mathcal{C}_2 \cap \mathcal{C}'_1 = \{0\}$ . It is also easy to see that  $\mathcal{C}_1 \cap \mathcal{C}_2 = \{0\}$ . Further, the quotient  $\mathcal{U}/\mathcal{C}_2$  cannot contain all of the extremal states of  $\mathcal{C}''$ , since  $\mathcal{U}/\mathcal{C}_2$  is a twisted topological Verma module, which already contains all of the extremal states of  $\mathcal{C}'_1$ . Therefore,  $\mathcal{C}_2 \cap \mathcal{C}'' = \mathcal{C}'_2 \neq \{0\}$  and the quotient  $\mathcal{U}/\mathcal{C}_2$  contains the submodule  $\mathcal{C}''/\mathcal{C}'_2$ . However, this can happen only in the case where  $\mathcal{C}''/\mathcal{C}'_2 = \{0\}$  or, equivalently,  $\mathcal{C}'' \subset \mathcal{C}_2$ , from which iii) follows.

To complete the proof, it remains to consider case b). Let  $\mathcal{U}' \subset \mathcal{U}$  be a massive submodule such that  $|x\rangle$  is the charged singular vector in  $\mathcal{U}'$ . This means that there exists a state  $|z\rangle$  with an admissible bigrading with respect to  $|x\rangle$  such that  $|x\rangle$  is a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of  $|z\rangle$ , whereas  $|z\rangle$  is not a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of  $|x\rangle$ . Since  $|z\rangle$  has an admissible bigrading with respect to  $|x\rangle$  as well as with respect to  $|y\rangle$ , and, also,  $(\ell_x, h_x) = (\ell_y + \theta_y, h_y - 1)$ , the state  $|z\rangle$  can be represented in the form  $|z\rangle = |w\rangle + a|u\rangle$ , where  $|u\rangle$  is a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of  $|y\rangle$  and  $a \in \mathbb{C}$ . Further,  $|w\rangle = 0$  in the quotient  $\mathcal{U}/\mathcal{C}_1$ , since otherwise we are in contradiction with the structure of the twisted topological Verma module  $\mathcal{U}/\mathcal{C}_1$ . Thus, we see that either  $|w\rangle \in \mathcal{C}_1$  or  $|w\rangle = 0$ . However,  $\mathcal{C}_1$  cannot contain all of the bigradings that are admissible with respect to  $|x\rangle$ , therefore there exists  $|z'\rangle \in \mathcal{C}''$  such that it is a dense  $\mathcal{G}/\mathcal{Q}$ -descendant of  $|z\rangle$ . We now see that  $\mathcal{U}' \cap \mathcal{C}'' \neq \{0\}$  or, equivalently, there exists  $\bar{\mathcal{C}} \subseteq \mathcal{C}''$  such that  $\bar{\mathcal{C}} \subset \mathcal{U}'$ , where the embedding is given by a charged singular vector. From part ii) of the Lemma, it follows that there exists  $\mathcal{C}_2 \subset \mathcal{U}$ , where  $\mathcal{C}_2$  is a submodule generated from a charged singular vector in  $\mathcal{U}$ ,  $\mathcal{C}_2$  is maximal ( $\mathcal{U} \supset \mathcal{C}''' \supset \mathcal{C}_2 \implies \mathcal{C}''' = \mathcal{C}_2$ ) and  $\bar{\mathcal{C}} \subset \mathcal{U}' \cap \mathcal{C}_2$ . We now have  $\mathcal{C}_2 \cap \mathcal{C}'' \neq \{0\}$ , which allows us to repeat the arguments regarding taking the quotient and, thus, to obtain iii).

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**Note added.** The fact that the charged singular vectors do not generate the massive Verma modules was used in the recent paper [D2]. As we saw in Theorem 2.13, one can be considerably more precise by saying that this is a twisted topological Verma module with the twist parameter  $-n$ , where  $n$  labels the charged singular vector. Similarly with the statement of [D2] regarding the degenerate case with two linearly independent singular vectors in the same grade: as we have seen, such vectors generate a direct sum of two twisted topological Verma modules, which makes the “fermionic uncharged singular vectors” introduced in [D2] excessive. The conditions for the absence of subsingular vectors applied in that paper to the derivation of characters of unitary representations, are a particular case of conditions of Proposition 3.14.

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